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Three problems

My choice of three problems, ordered in increasing difficulty.
The first is elementary, but the last is a very difficult problem.
The three problems deal with relatively simple geometric configurations but the answers and the solutions are surprising, therefore they are challenging.

1. With two parallelograms

I is a point which does not belong to the sides of a parallelogram $ABCD$. IB intersects AD at E and CD at F. ID intersects AB at G and CB at H. J is the fourth vertex of the parallelogram $AGJE$. Show that J lays on the line IC.

This problem is elementary but without any other indication it is challenging, even for a good secondary student.
An elementary solution requires only Thales, but there is a simpler solution using homotheties.

2. The bicycle's wheel

Find the locus of the center of a bicycle wheel touching the floor and two corner walls.

I just like this problem and it is not very difficult to solve.
If it is easy to guess the perimeter of the locus, the locus itself is less obvious.
Remark: I did not find a reasonably simple and short proof of the converse; nevertheless we may admit the result by continuity.

3. Sixteen centers of incircles and excircles

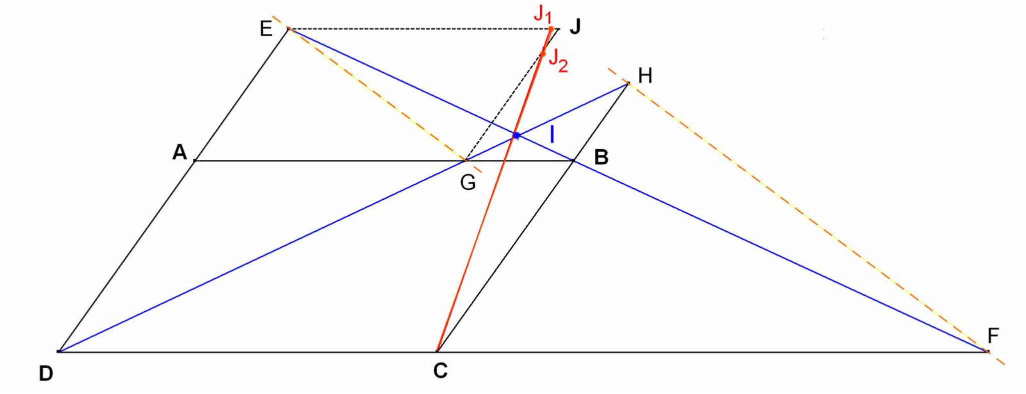
(women's agrégation - France - 1926)

Four concyclic points define four triangles; what is the configuration of the sixteen centers of their incircles and excircles?

This is a beautiful but very difficult problem. This surprising configuration is easy to find... after tedious constructions! Using the perpendicularity of the bisectors of an angle makes the constructions easier and gives an indication for the solution.
The proof does not need sophisticated tools, but it is nevertheless an arduous task to achieve it; the high number of points involved makes the reasoning tricky.

With two parallelograms

I is a point which does not belong to the sides of a parallelogram $ABCD$. IB intersects AD at E and CD at F. ID intersects AB at G and CB at H. J is the fourth vertex of the parallelogram $AGJE$. Show that J lays on the line IC.



1 With so many parallel lines, Thales is obviously involved. In fact no other tool is required. By Thales theorem, using parallel lines AD and BC (resp. AB and DC) with secants IB and ID, we get $\frac{IB}{IE} = \frac{IH}{ID}$ (resp. $\frac{IB}{IF} = \frac{IG}{ID}$).

From these equalities follows $IB \times ID = IE \times IH = IF \times IG$ and then $\frac{IE}{IF} = \frac{IG}{IH}$.

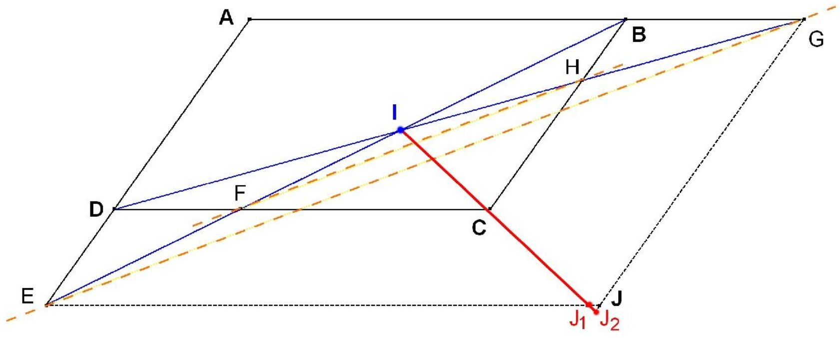
Now by the converse of Thales theorem we have EG and FH parallel.

Let J_1 (resp. J_2) be the intersection of IC with EJ (resp. GJ). Again by Thales theorem we get, using parallel lines BC and GJ with secants IC and IF, $\frac{IE}{IF} = \frac{IJ_1}{IC}$,

and using parallel lines DC and EJ with secants IC and IH, $\frac{IG}{IH} = \frac{IJ_2}{IC}$.

Taking in account that the two first fractions are equal, we get $IJ_1 = IJ_2$, which means $J_1 = J_2 = J$, and we are done: J lays on IC.

With I inside $ABCD$ the diagram looks "simpler".



2 A simpler but less elementary solution uses homothecies (J_1 and J_2 are unnecessary).

Let h_1 (resp. h_2) be the homothecy with center I which maps B on E and H on D (resp. F on B and D on G). $h = h_2 \circ h_1 = h_1 \circ h_2$ has I as center and maps H on G and F on E, thus the lines EG and FH are parallel.

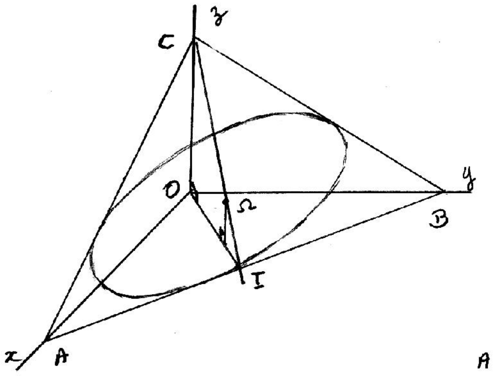
Now h maps HC on GJ and FC on EJ, therefore it maps C on J and we are done.

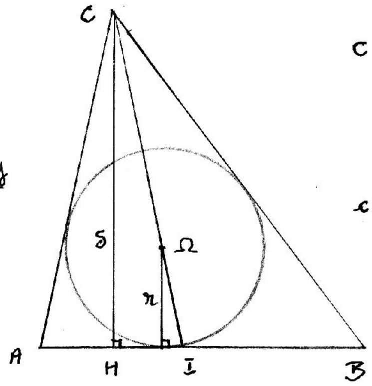
The bicycle's wheel

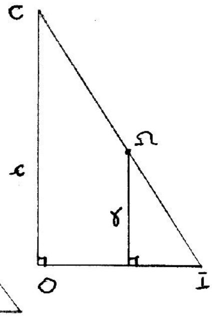
Find the locus of the center of a bicycle wheel touching the floor and two corner walls.

Let $(0,x,y,z)$ be a cartesian coordinate system, $\Omega(\alpha,\beta,\gamma)$ the center of the wheel, and r its radius. If the wheel remains tangent to one of the coordinate axes, the locus of its center is obviously a quarter circle in a plane perpendicular to that axis, at the distance r from O (see last diagram). All the points of the three quarter circles are the distance $r\sqrt{2}$ from O. Let us show that this always holds.

The wheel touches the coordinate planes as shown below and its plane intersects the axes at points $A(a,0,0)$, $B(0,b,0)$ and $C(0,0,c)$, where a,b,c are positive.







Let $px+qy+rz=abc$ be the equation of the plane (ABC). Expressing that the plane goes through A, B and C gives the conditions $abc=pa=qb=rc$, thus $p=bc$, $q=ac$, $r=ab$.

The distance from O to this plan is equal to $\dfrac{abc}{\sqrt{(bc)^2+(ca)^2+(ab)^2}}$.

Let I be the point where $C\Omega$ intersects AB, and $\delta = CH$ the distance from C to line AB.

By Thales in the triangles CHI and COI we get $\dfrac{r}{\delta}=\dfrac{I\Omega}{IC}=\dfrac{\gamma}{c}$, thus $\delta=\dfrac{rc}{\gamma}$

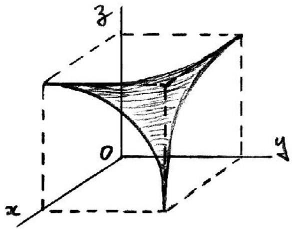
The volume of the tetrahedron $OABC$, $V=\dfrac{1}{3}\times\dfrac{ab}{2}\times c=\dfrac{abc}{6}$, is also

$$V=\frac{1}{3}\times(\frac{1}{2}\times\frac{rc}{\gamma}\times\sqrt{a^2+b^2})\times\frac{abc}{\sqrt{(bc)^2+(ca)^2+(ab)^2}} \quad \text{thus} \quad \gamma=\frac{rc\sqrt{a^2+b^2}}{\sqrt{(bc)^2+(ca)^2+(ab)^2}}.$$

Now we have $\gamma^2=\dfrac{r^2\times c^2(a^2+b^2)}{(bc)^2+(ca)^2+(ab)^2}$, and the same holds for β^2 and α^2.

Then $O\Omega^2=\alpha^2+\beta^2+\gamma^2=2r^2$ and Ω is on the sphere with center O and radius $r\sqrt{2}$.

All coordinates of Ω are less than or equal to r, thus the locus is included in the cube with edge r constructed on the coordinate axes (one vertex at O and three edges on the axes).
Finally the locus of the wheel's center is the part of the sphere limited by three quarter circles with radius r.



Remark: I did not find a reasonably simple and short proof of the converse; nevertheless we may admit the result by continuity (every point of the sphere and inside the cube is center of a circle tangent to the coordinate planes).

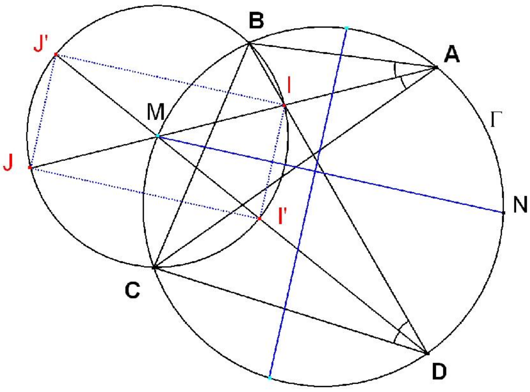
Sixteen centers of incircles and excircles

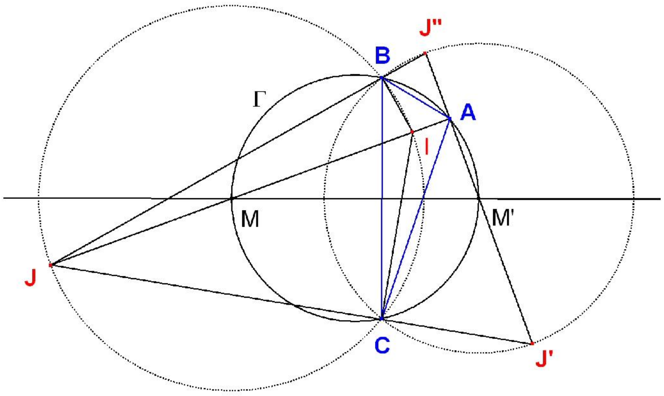
Four concyclic points define four triangles; what is the configuration of the sixteen centers of their incircles and excircles?

Lemma 1: *Let A, B, C and D be four points on a circle with center O.*
Then there are two perpendicular lines Δ and Δ' which verify the property:
For any couple of arcs with distinct ends like $\overset{\frown}{BC}$ and $\overset{\frown}{AD}$ with respective midpoints M and N, the line MN is parallel to Δ or Δ'.

<u>Proof</u> (left diagram): In the complex plane with origin O the length unit is the radius of Γ, and we denote a,b,c,d the arguments of A,B,C,D respectively. Then, $(mod\,\pi)$, the argument of N (resp. M) is $\dfrac{a+d}{2}$ (resp. $\dfrac{b+c}{2}$), and taking in account that $e^{i\frac{a+d}{2}}-e^{i\frac{b+c}{2}}=e^{i\frac{a+b+c+d}{4}}[e^{i\frac{a+b-c-d}{4}}-e^{-i\frac{a+b-c-d}{4}}]$, the argument of $\overrightarrow{MN}$ is $\dfrac{a+b+c+d}{4}$ at $n\dfrac{\pi}{2}$ near.

Let Δ (resp. Δ') be the line with argument $\dfrac{a+b+c+d}{4}+k\dfrac{\pi}{2}$ where k is even (resp. odd). This definition is independent from the choice of the two arcs above, and one of these lines is collinear to $\overrightarrow{MN}$. The same holds for the other couples of arcs and their midpoints.





Lemma 2: *Let I and J be the centers of two circles tangent to the sides of a triangle ABC with circumcircle Γ. Then the line IJ goes through one vertex (for example A) and intersects Γ again at a point M equidistant to I, J, B and C.*

<u>Proof</u> (right diagram): Two of the three sides of triangle ABC are common tangents of same nature to the circles with centers I and J, thus they intersect at a point of IJ, and this point is therefore one of the vertices of ABC (point A on the diagram).

Moreover IJ is one of the bisectors of $\angle ABC$ and M is the midpoint of an arc $\overset{\frown}{BC}$ (inscribed angle). Thus M lays on the perpendicular bisector D of segment BC, and by definition on IJ.

Let M_1 be the midpoint of IJ. Knowing that $\angle IBJ$ and $\angle ICJ$ are right angles, we have the equidistance of M_1 to I, J, B and C, hence M_1 lays on D. Recalling that M_1 lays on IJ, two cases are possible:

a) if $IJ \neq D$, we have $M = IJ \cap D = M_1$, thus M is the midpoint of IJ,

b) if $IJ = D$, this line is the interior bisector of $\angle ABC$ and
$\angle MBI = \angle MBC + \angle CBI = \angle MAC + \angle ABI$ (inscribed angle) $= \angle BAI + \angle ABI = \angle MIB$

(exterior angle). Hence the triangle BMI is isosceles and similarly for BMJ because IBJ is rectangular.
Finally $MI = MB = MJ$, and the same holds for C.

Corollary: *A, B, C and D being four concyclic points, the midpoint M of an arc $\overset{\frown}{BC}$ is the center of a rectangle whose:*
- vertices are the centers of circles tangent to the three sides of the triangles ABC and DCB,
- diagonals lay on the lines AM and DM,
- sides are parallel to Δ and Δ'.

Proof (left diagram): Let I and J (resp. I' and J') be the centers laying on AM (resp. DM). By lemma 2 the point M is equidistant from the six points I,J,B,C,I',J', thus we have the rectangle, its vertices and its diagonals.
At last, if N is the midpoint of an arc $\overset{\frown}{AD}$, the line MN is a bisector of $\angle AMD$ (inscribed angle) and is, by lemma 1, parallel to Δ or Δ'.

Conclusion: Let us focus on the subset of the incenter and excenters of one of the triangles, for example C_d for ABC.
Trough $I \in C_d$ let us draw Δ_I (resp. $\Delta_{I'}$) parallel to Δ (resp. Δ'). I being the common point of three bisectors of the triangle, by associating each of them with Δ_I and $\Delta_{I'}$, we get three rectangles such as the one defined in the corollary.
Among the vertices of these rectangles, three points, one from C_a, one from C_b and one from C_c come then on Δ_I; the same happens on Δ'_I. Therefore we get four lines parallel to Δ, with four points, one from each subset, on each line, and the same with Δ'.

Finally the sixteen points belong to the perimeters of two rectangles with parallel sides:
- one, inside Γ, has the incenters as vertices,
- the other, outside Γ, has four excenters as vertices (the ones which are on the interior bisectors of the convex quadrilateral) and the remaining eight excenters are the intersections of the sides of the two rectangles.

Remark:
The two rectangles become squares if and only if the convex quadrilateral is itself a square or has one diagonal as axis of symmetry. Each mirror symmetry of the quadrilateral is preserved on the whole diagram.