# A Group Theoretic Approach to Kaleidocycles and Cubeocycles

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**Abstract**

In this paper, we further Doris Schattschneider's work on kaleidocycles, which are three-dimensional rings made from chains of 2*n* regular tetrahedra attached at edges. We show that the symmetry group of a kaleidocycle is $D_n \oplus Z_2 \oplus Z_2$. In addition, we extend kaleidocycles to cubeocycles (pronounced "cube-o-cycles"), three-dimensional rings made from chains of 2*n* cubes attached at antipodal vertices, and show that the symmetry group of a cubeocycle is $D_n \oplus Z_2 \oplus D_3$.

## 1. Introduction

In everyday life, we find various examples of uses and implementations of the concept of symmetry. When we visit museums, whether they are art or history museums, we can appreciate the symmetries in a painting, a sculpture, the architecture, or the fossils of ancient animals and plants. The human eye is naturally drawn to objects that appear to be symmetric because symmetric objects and figures are perceived to be beautiful or appealing. Even the human body has a number of symmetries that people believe to be pleasing and attractive. However, the concept of symmetry is not so easily defined because of the numerous notions of symmetries that people possess. Two popular notions of symmetry are "well-balanced" or "well-proportioned" and "bilateral" symmetry; see [5] for further discussion of the concept of symmetry.

This concept has been the focus of many questions in the intersection of mathematics and art. For example, take wallpaper, which by definition has at least a translational symmetry. Can we find other symmetries in wallpapers? What ones and how do they arise? We refer the reader to [1] for more information on wallpapers. Closely connected with the subject of wallpaper is the work of the artist M.C. Escher and many papers have been written about his study of symmetry. For a more in depth discussion of these symmetries, see [4].

In this paper, we take an algebraic approach to symmetries and focus our attention on the symmetries that arise in three-dimensional objects called *kaleidocycles* and *cubeocycles*. We present the definitions of these terms in later sections. We can translate the geometrical problem of finding the symmetries of an object into an algebraic problem of finding the elements that compose the symmetry group of that object. To begin our endeavor we define some algebraic concepts that will guide us through our description of the groups.

Algebraically, we define a symmetry in terms of a function with the special characteristic of distance preservation.

**Definition 1.1:** Let *f* be a function from $R^n$ to $R^n$. Then, *f* is an *isometry* if it preserves distance; that is, $d(a,b) = d(f(a), f(b))$ for any *a* and *b* in $R^n$.

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**Example 1.2:** Let $f(x) = -x$ be a function on *R*. Then, $d(a,b) = |a-b|$. Also, $d(f(a), f(b)) = |-a+b| = |-1(a-b)| = |a-b|$. Since, $d(a,b) = d(f(a), f(b))$, $f(x)$ is an isometry.

We are ready to define the term "symmetry" as it will be used throughout this paper.

**Definition 1.3:** Let *F* be a subset of $R^n$. Then the *symmetries* of *F* are the isometries of *F* onto *F*.

In other words, the symmetries of *F* are going to be all the functions that preserve the shape and size of the figure within the same configuration.

**Example 1.4:** A regular octagon has a number of symmetries. It has eight rotations obtained by rotating the figure in counter-clockwise 45º increments about the center of the octagon. Also, there are eight reflections, where half of the axes of reflection are obtained by joining opposite vertices and the other half by joining midpoints of opposite sides. If we allow *r* to be one counter-clockwise rotation by 45º and allow *f* to be a reflection along one specified axis of the octagon, then every possible symmetry can be obtained through a composition of powers of *r* and *f*. In Figure 1.5, we show the octagon (labeled *e*), one rotation of 45º (labeled *r*), a vertical reflection (denoted by *f*), and the composition of one rotation of 45º and one vertical reflection (labeled *fr*).

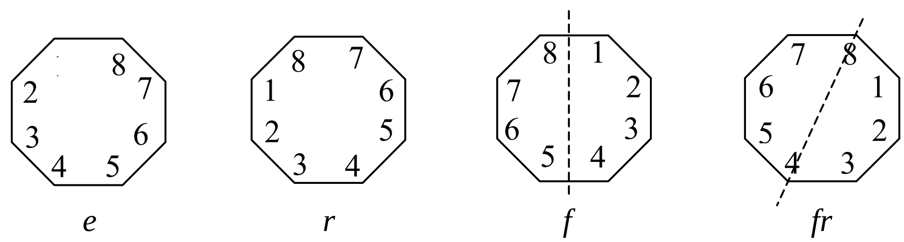


Figure 1.5.

Up to this point we have seen how we can define the actions on a polygon in terms of functions, now we go one step further. When we are describing the symmetries of a specific figure or object, the set of symmetries of the object always form a group. Depending on the type of symmetries that we find, this group may be a specific case of one of the following.

**Definition 1.6:** Let $\pi_n$ be a regular *n*-gon. The *dihedral group* of $\pi_n$ is the set of all of the symmetries of $\pi_n$; we denote the dihedral group by $D_n$.

**Example 1.7:** The set of all symmetries of the regular octagon described in Example 1.5 forms the group $D_8$. The group $D_8 = \{e, r, r^2, r^3, r^4, r^5, r^6, r^7, f, rf, r^2 f, r^3 f, r^4 f, r^5 f, r^6 f, r^7 f\}$ which is the group generated by *r* and *f*, subject to the relations $r^8 = f^2 = e$ and $frf = r^7$.

**Definition 1.8:** Let *G* be a group. *G* is *cyclic* if there exists an *a* in *G* such that for any *b* in *G*, $b = a^n$. We call *a* the *generator* of G.

It is known that a finite cyclic group with *n* elements is isomorphic to $Z_n$. For an example of a cyclic group, consider the set of rotations of $D_8$. The set of rotations of $D_8$ forms a group under the operation of composition. The group of rotations is a cyclic group because *r* is the generator since any element in the group of rotations can be obtained as a power of *r*. Besides the more well-known dihedral and cyclic groups, we will need three other groups to characterize the symmetry groups of the objects we will encounter in this paper.

**Definition 1.9:** We define the *symmetric group of degree n*, denoted $S_n$, to be the set of all permutations of {1, 2, …, *n*}.

**Definition 1.10:** We call *s* in $S_n$ an *even* permutation if *s* can be factored into an even number of two-cycles. The *alternating group of degree n*, denoted $A_n$, is the group of even permutations in $S_n$.

**Definition 1.11:** The *Klein-4 group*, denoted $K_4$, is a subgroup of $S_n$ for $n \geq 4$ containing the permutations (1), (12), (34), and (12)(34). We can also think of $K_4$ as isomorphic to a group containing the following symmetries: the identity, one 180° rotation, and two perpendicular reflections.

**Example 1.12:** The symmetric group $S_4$ is the group of all permutations of the four elements: 1, 2, 3, and 4. Thus, the permutation $\sigma = (123)$ is an element of $S_4$. In addition, $\sigma$ is an even permutation because it can be expressed as an even number of 2-cycles as follows $\sigma = (13)(12)$. Thus, $\sigma$ is also an element of $A_4$.

**Definition 1.13:** Let *G* and *H* be groups. The *direct sum* is defined to be $G \oplus H = \{(g,h) | g \in G, h \in H\}$ under coordinate-wise operation. Note that elements may be viewed as *gh*.

We have defined all of the algebraic terms needed in this paper. More definitions will arise as we move through the characterization of different three-dimensional objects. We begin with an object that most people have seen before: a tetrahedron. Then, we will move through the characterizations of the symmetry groups discovered when two or more tetrahedra are attached in various ways until we characterize the symmetry group of a kaleidocycle.

## 2. Tetrahedra

A *regular tetrahedron* is a three-dimensional object composed of four copies of an equilateral triangle such that the vertices of three triangles intersect in a single vertex; see Figure 2.1.

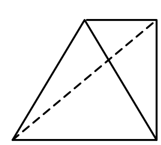

Figure 2.1.

Notice that a regular tetrahedron has a number of symmetries. One symmetry is the identity rotation. The nontrivial rotations of a regular tetrahedron are determined by different axes of rotation: four axes are through a vertex and the center of the opposite triangle (each such axis yields two rotations of 120° increments); three axes are through midpoints of two opposite edges (each such axis yields one rotation of 180°). See page 105 of [2] for pictures of these rotational axes. We have shown that the set of rotational symmetries of the tetrahedron is isomorphic to the group $A_4$, which has order 12. Besides rotations, a regular tetrahedron also has reflections. The planes of reflection contain one edge and bisect the tetrahedron. A regular tetrahedron has six planes of reflection. Writing the symmetries obtained through rotation and reflection as permutations and composing the various permutations produce permutations that are neither reflections nor rotations, which are additional symmetries of the tetrahedron. Specifically, composing three reflections yields six additional symmetries of a regular tetrahedron. These additional symmetries are neither rotations nor reflections because they require an inversion of the vertices of the tetrahedron which cannot be obtained by rotating or reflecting the tetrahedron; see Figure 2.2.

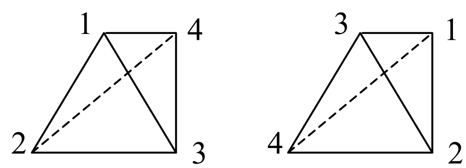


Figure 2.2.

The object formed by the symmetry in Figure 2.2 is the composition of a reflection along the axis containing vertices 3 and 4 and the rotation through the axis though the midpoint of the line containing 1 and 4 and the line containing 2 and 3; however, it could not be obtained by simply rotating or reflecting the tetrahedron. The set of symmetries formed by the rotations, reflections, and those that are neither form the group $S_4$, which has order 24.

The objects obtained by attaching two regular tetrahedra in various ways also have a number of symmetries. Two regular tetrahedra attached on one face form an object with two antipodal vertices with an equatorial equilateral triangle; see Figure 2.4. This object has an axis of rotation through the two antipodal vertices (which produces two rotations of 120° increments). This object has three other axes of rotation through the equator that are each given by a non-antipodal vertex and the midpoint of the opposite edge (each rotates the object 180°). Thus, considering the identity as a rotation, the object has six rotations. The object also has four planes of reflection that are each given by three vertices and bisect the object. Two additional symmetries that are neither rotations nor reflections are obtained by composing the permutations of the second type of rotations and the reflections. We show, below, that the set of symmetries of two regular tetrahedra attached on a face is isomorphic to the group $D_3 \oplus Z_2$.

**Lemma 2.3:** *The group G of symmetries of the two regular tetrahedra attached on one face is isomorphic to* $D_3 \oplus Z_2$.

**Proof:** Let *G* be the group of symmetries of two regular tetrahedra attached on one face as seen in Figure 2.4.

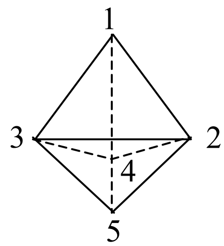


Figure 2.4.

Then the elements of $G$, in permutation form, are (1), (234), (243), (15)(34), (15)(24), (15)(23), (24), (23), (34), (15), (243)(15), and (234)(15). Define a function $\varphi$ from $G$ to $D_3 \oplus Z_2$ that maps $\alpha$ to $(\alpha,0)$ and maps $\alpha(15)$ to $(\alpha,1)$. We assume that (15) is not a factor of $\alpha$ so the elements that correspond to $\alpha$ are (1), (234), (243), (24), (23), and (34). The other elements correspond to $\alpha(15)$. We must show that $\varphi$ is one-to-one and onto. Assume $\varphi(\alpha) = \varphi(\beta)$. Then, $(\alpha,0) = (\beta,0)$. Thus, $\alpha = \beta$. If we assume that $\varphi(\alpha(15)) = \varphi(\beta(15))$, then $\alpha = \beta$. However, $\varphi(\alpha) \neq \varphi(\beta(15))$ because $\alpha$ will be mapped to $(\alpha,0)$ and $\beta(15)$ will be mapped to $(\beta,1)$ and $\alpha \neq \beta$. Let $(\delta, n)$ be an element of $D_3 \oplus Z_2$. If $n = 0$, then $\delta = \alpha$, and if $n = 1$, then $\delta = \alpha(15)$. Thus, $\varphi$ is one-to-one and onto.

Now, we must show that $\varphi(\sigma\tau) = \varphi(\sigma)\varphi(\tau)$ for all $\sigma, \tau$ in G. We have three cases:

**Case 1:** $\sigma = \alpha, \tau = \beta$

$\varphi(\sigma\tau) = \varphi(\alpha\beta) = (\alpha\beta,0)$ and $\varphi(\sigma)\varphi(\tau) = \varphi(\alpha)\varphi(\beta) = (\alpha,0)(\beta,0) = (\alpha\beta,0)$

**Case 2:** $\sigma = \alpha(15), \tau = \beta$

$\varphi(\sigma\tau) = \varphi(\alpha(15)\beta) = (\alpha\beta,1)$ and $\varphi(\sigma)\varphi(\tau) = \varphi(\alpha(15))\varphi(\beta) = (\alpha,1)(\beta,0) = (\alpha\beta,1)$

**Case 3:** $\sigma = \alpha(15), \tau = \beta(15)$

$\varphi(\sigma\tau) = \varphi(\alpha(15)\beta(15)) = (\alpha\beta,0)$ and $\varphi(\sigma)\varphi(\tau) = \varphi(\alpha(15))\varphi(\beta(15)) = (\alpha,1)(\beta,1) = (\alpha\beta,0)$

Thus, $G$ is isomorphic to $D_3 \oplus Z_2$.■

The object formed by attaching two regular tetrahedra at one vertex such that the edges at the two ends form three planes and the centers of the two tetrahedra are on one line through the attached vertex also has the same number of symmetries as the previous object; see Figure 2.5.

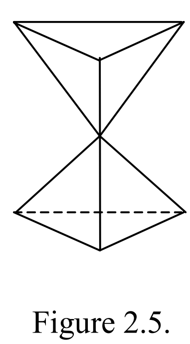

Figure 2.5.

One axis of rotation, denoted by *l*, contains the centers of the two base equilateral triangles and the vertex at which the two tetrahedra are attached, and this axis produces two rotations of 120° increments. Three other axes of rotation are lines perpendicular to *l*. These axes of rotation produce three rotations of 180°. One plane of reflection contains these three axes of rotation and cuts the object into the two tetrahedra. Three other planes of reflection each contain the vertex at which the two tetrahedra are attached and two other vertices such that the plane bisects both tetrahedra. Composing the rotations of the first type and the reflections of the first type yields two additional symmetries that are neither reflections nor rotations. Thus, this object also has twelve symmetries including the identity, and by an argument similar to that of Lemma 2.3, the group of symmetries of two regular tetrahedra attached at a vertex such that the edges of the two tetrahedra form three planes is isomorphic to $D_3 \oplus Z_2$. Moreover, we show that the group $D_6$ is isomorphic to $D_3 \oplus Z_2$.

**Lemma 2.6:** *The group $D_6$ is isomorphic to $D_3 \oplus Z_2$.*

**Proof:** Define a function $\varphi$ from $D_6$ to $D_3 \oplus Z_2$ that maps *r* to $(R,1)$ and *f* to $(F,1)$ where *r* is a rotation of 60º in $D_6$, *f* is a vertical flip in $D_6$, *R* is a rotation of 120º in $D_3$, and *F* is a vertical flip in $D_3$. To prove this lemma, we must show that $\varphi(r^m f^n r^k f^q) = \varphi(r^m f^n)\varphi(r^k f^q)$ for all *r*, *f* in $D_6$. The map $\varphi$ is one-to-one and onto by inspection—it is a direct correspondence on generators.

Then, we have three cases to show the operation preservation.

**Case 1:** $n = 0$, $q = 0$

$$\varphi(r^m r^k) = \varphi(r^{m+k}) = (R^{m+k}, m+k(\text{mod}\,2))$$
$$\varphi(r^m)\varphi(r^k) = (R^m, m)(R^k, k) = (R^m R^k, m+k(\text{mod}\,2)) = (R^{m+k}, m+k(\text{mod}\,2))$$

**Case 2:** $n = 1$, $q = 0$

$$\varphi(r^m r^k f) = \varphi(r^{m+k} f) = (R^{m+k}F, m+k+1(\text{mod}\,2))$$
$$\varphi(r^m)\varphi(r^k f) = (R^m, m)(R^k F, k+1) = (R^m R^k F, m+k+1(\text{mod}\,2)) = (R^{m+k}F, m+k+1(\text{mod}\,2))$$

**Case 3:** $n = 1$, $q = 1$

$$\varphi(r^m f r^k f) = \varphi(r^m (frf^{-1})^k) = \varphi(r^m (r^5)^k) = \varphi(r^{m+5k}) = (R^{m+5k}, m+5k(\text{mod}\,2))$$
$$\varphi(r^m f)\varphi(r^k f) = (R^m F, m+1(\text{mod}\,2))(R^k F, k+1(\text{mod}\,2)) = (R^m F R^k F, m+k+2(\text{mod}\,2))$$
$$= (R^m (FRF^{-1})^k, m+k(\text{mod}\,2)) = (R^m (R^2)^k, m+k(\text{mod}\,2)) = (R^{m+2k}, m+k(\text{mod}\,2))$$

Now, we must show that $m+5k = m+2k(\text{mod}\,3)$ and that $m+5k = m+k(\text{mod}\,2)$. In $D_3$, $m+2k+3k(\text{mod}\,3) = m+5k$ because the addition of 3*k* to the left side will only add zero to the left side because 3*k*(mod 3) equals zero. In addition, $m+k(\text{mod}\,2) = m+5k$ because adding 4*k* to the left side will add zero to the left side because 4*k*(mod 2) equals zero.

Thus, $D_6$ is isomorphic to $D_3 \oplus Z_2$.■

Thus, the group of symmetries of two regular tetrahedra attached on a face and the group of symmetries of two regular tetrahedra attached at a vertex so that the edges of the two tetrahedra form three planes are both isomorphic to $D_6$.

The object formed by attaching two tetrahedra at one vertex as before, except ensuring that the edges of the two triangles on the end of the object form no planes, has symmetries; see Figure 2.7.

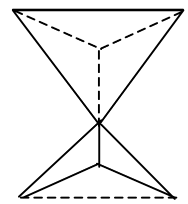

Figure 2.7.

One axis of rotation is through the centers of the two triangles on the ends of the object and the vertex at which the two tetrahedra are attached, and this axis produces two rotations of 120° increments. Three other axes of rotation are through the vertex at which the two tetrahedra are attached and perpendicular to the other axis of rotation, where each axis produces one rotation of 180°. Therefore, including the identity as a rotation, this object has six rotations. This object has no reflections or symmetries that are neither rotations nor reflections, so the set of symmetries of this object is $Z_6$.

Another way to attach two tetrahedra is along an edge; see Figure 2.8.

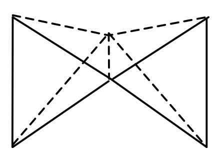

Figure 2.8.

This object yields eight symmetries. One axis of rotation is from the midpoints of the two outer edges of the object to the midpoint of the connected edge where the axis produces one rotation of 180°. Another axis of 180° rotation is given by the connecting edge. One plane of reflection is along the connecting edge and dividing the object into two tetrahedra. Another plane of reflection is given by the four vertices of the outer edges. In addition, another plane of reflection contains the midpoints of the outer edges of the object and the connected edge. By composing the permutations of the reflections, we obtain two symmetries that are neither rotations nor reflections. Therefore, we show below that this object has eight symmetries and its group of symmetries is isomorphic to $K_4 \oplus Z_2$.

The next step in finding the symmetries of attached tetrahedra is to consider the various ways to attach three tetrahedra such that the object produces symmetries. Attaching three tetrahedra at a vertex such that the three tetrahedra are connected at two edges produces an object that has symmetries; see Figure 2.10.

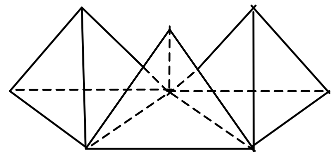

Figure 2.10.

The symmetries of the object consist of one plane of reflection through the connected and middle vertex of the center tetrahedron and the identity. Similarly, another object of three tetrahedra that has symmetries is obtained by attaching an additional tetrahedron at the center vertex of Figure 2.5 such that six vertices lie in a single plane and the line containing the center of the additional tetrahedron is perpendicular to the line containing the centers of the other two tetrahedra; see Figure 2.11.

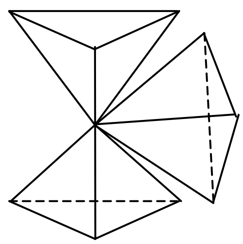

Figure 2.11.

This object yields two symmetries: one plane of reflection through the attached vertex and the midpoint and one vertex of the additional tetrahedra and the identity.

While there are many more ways to attach three tetrahedra, two particularly interesting ways that have more than two symmetries exist. One way to attach three tetrahedra is to attach two tetrahedra on a face and then attach another tetrahedron at one of the vertices of the two attached tetrahedra such that no other edges or faces of the two attached tetrahedra touch the additional tetrahedron and the face opposite to the attached vertex is parallel to the attached face; see Figure 2.12.

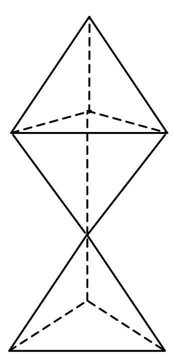

Figure 2.12.

This object has one axis of rotation through the attached vertex and the vertex on the end of the object (which yields two rotations of 120° increments), the identity, and three planes of reflection through the four vertices that are coplanar. Thus, the object has six symmetries which form the group $D_3$. The other interesting way to attach three regular tetrahedra is to attach three tetrahedra along one edge so that one face of each tetrahedron is attached to one face of one other tetrahedron, see Figure 2.13.

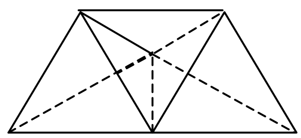

Figure 2.13.

This object has one axis of rotation through the midpoint of the top edge and the midpoint of the connected edge (the axis rotates the object 180°). This object has two planes of reflection through the two centers of the object that are perpendicular to each other (each plane yields one symmetry). Thus, since the object has two perpendicular reflections and a 180° rotation, the set of symmetries of the object is the group $K_4$.

## 3. Kaleidocycles

The next question, after discovering how three tetrahedra can be attached so that the object has symmetries, is how four regular tetrahedra can be attached. The most interesting possibility to attempt is that four tetrahedra could be attached in a row (or, rather, in a cycle) along edges so that the first and the fourth also connect along an edge, or in a *kaleidocycle*. For a more in-depth discussion of kaleidocycles, refer to [3].

**Definition 3.1:** A *kaleidocycle* is a three-dimensional ring made from a chain of regular tetrahedra attached at edges.

Folding a piece of paper with a grid of equilateral triangles, as seen in Figure 3.2, by the following directions will form a kaleidocycle.

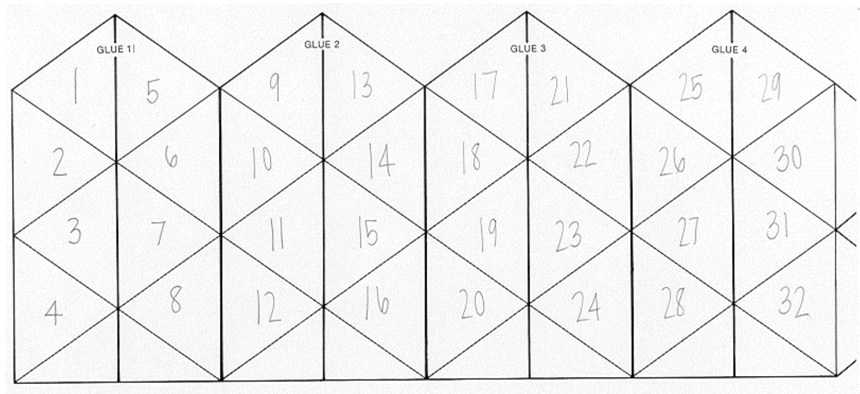


Figure 3.2.

By folding the vertical lines face-to-face and folding back-to-back on all diagonal lines, the pattern will begin to take shape. Curling the bottom triangles around to meet the tabs at the top of the pattern and gluing these top and bottom tabs together exactly will form a chain of eight linked tetrahedra. Bringing the ends together and gluing the two ends of the chain together will form the kaleidocycle.

However, this object can0not be constructed with four tetrahedra because attaching two tetrahedra along an edge creates an object with a central angle of 120° and attaching two of these objects only creates an object with a central angle of 240°. An object created with tetrahedra in a cycle must contain at least six tetrahedra in order to produce an object with the needed central angle of 360° to get the first tetrahedron to be attached to the sixth.

We can do what is described above for eight regular tetrahedra so that eight regular tetrahedra are attached in a cycle so that the first and the eighth are also connected along an edge; the resulting object is a *square kaleidocycle*; see Figure 3.3.

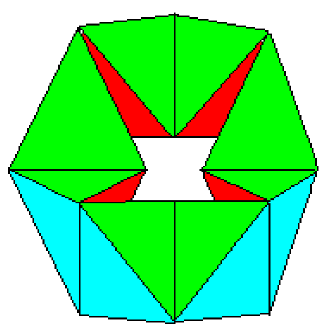

Figure 3.3.

The grid used above in Figure 3.2 is the grid that we use to assist us in finding the symmetries of the square kaleidocycle. For our purposes, we numbered each triangle in the grid and noted the symmetry by the permutation of triangles. The numbers on the triangles correspond to the faces that we use in the cycle notation of the symmetries. The square kaleidocycle has a number of interesting symmetries including rotations, reflections, a *writhe* (pushing the internal vertices in and pulling the external vertices up to obtain the same object), and a number of others which are neither rotations or reflections. The square kaleidocycle has one axis of rotation that is a vertical line through the center hole of the object (yielding four rotations of 90° increments). The four other rotations are determined by the axes through opposite edges (each such axis yields one rotation of 180° where the entire kaleidocycle flips over). Four of the planes of reflection are given by planes containing opposite edges (each plane yields one symmetry). In addition, the square kaleidocycle has a horizontal reflection through a plane containing all of the attached edges. Rotating the kaleidocycle and then horizontally reflecting the object produces three elements that are neither rotations nor reflections. These rotations, reflections, the three "other" symmetries along with the identity produce 16 symmetries of the object. We will show in Lemma 3.5 that these 16 symmetries form the group $D_4 \oplus Z_2$. First, we need a computational tool.

**Lemma 3.4:** *If q is a horizontal reflection,* $\alpha$ *is a rotation, and* $\beta$ *is a flip of a kaleidocycle,* $\alpha(q)\beta(q) = \alpha\beta(q)(q) = \alpha\beta$.

**Proof:** If we rotate a kaleidocycle and then reflect it horizontally, it is the same as a horizontal reflection followed by a rotation. If we flip a kaleidocycle and then reflect it horizontally, it is the same as a horizontal reflection followed by a flip. Thus, the horizontal reflections commute, and $\alpha(q)\beta(q) = \alpha\beta$ .■

**Lemma 3.5:** *The group of previously described 16 symmetries of the square kaleidocycle is isomorphic to the group* $D_4 \oplus Z_2$.

**Proof:** Let *G* be the group of 16 symmetries of the square kaleidocycle that were previously described. In permutation notation, these elements are provided in Appendix A. Define a function $\varphi$ from these elements to $D_4 \oplus Z_2$ that maps $\alpha$ to $(\alpha,0)$ and maps $\alpha(1,3)(5,7)(9,11)(13,15)(17,19)(21,23)(25,27)(29,31)$ to $(\alpha,1)$. The specific numbers arise from the numbers in the grid of Figure 5.2. For example, (1, 3) would move face 1 to face 3 and face 3 to face 1. The proof of Lemma 2.3 with *q* = (1, 3) (5, 7) (9, 11) (13, 15) (17, 19) (21, 23) (25, 27) (29, 31) (*q* is a horizontal reflection) in the place of (15) shows that $\varphi$ is one-to-one and onto.

From there, we must show that $\varphi(\sigma\tau) = \varphi(\sigma)\varphi(\tau)$ for all $\sigma, \tau$ in G. Again, we have the same three cases as the proof of Lemma 2.3 with the substitution of *q* as (15). The first two cases correspond to case 1 and case 2 of Lemma 2.3.

**Case 3:** $\sigma = \alpha(\mathrm{q})$, $\tau = \beta(q)$.

$\varphi(\sigma\tau) = \varphi(\alpha(q)\beta(q)) = (\alpha\beta,0)$ by Lemma 3.4 and $\varphi(\sigma)\varphi(\tau) = \varphi(\alpha(q))\varphi(\beta(q))$

$= (\alpha,1)(\beta,1) = (\alpha\beta,0)$

Therefore, *G* is isomorphic to $D_4 \oplus Z_2$ .■

The square kaleidocycle also has a writhing symmetry where the object is turned in on itself. Composing these 16 elements with the writhing symmetry (denoted in permutation notation as (1, 3) (2, 4) (5, 7) (6, 8) (9, 11) (10, 12) (13, 15) (14, 16) (17, 19) (18, 20) (21, 23) (22, 24) (25, 27) (26, 28) (29, 31) (30, 32) from the grid of Figure 5.2) of the square kaleidocycle produces another 16 symmetries of the object providing a total of 32 symmetries for the square kaleidocycle. Thus, the symmetry group for the symmetries of the square kaleidocycle is $D_4 \oplus Z_2 \oplus Z_2$.

We can find the subgroups of the square kaleidocycle by coloring the kaleidocycle in such a way that certain symmetries of the square kaleidocycle are not present. Numerous ways to color the kaleidocycles to find the subgroups exist, and the ones we use here are not the only ways to color a square kaleidocycle to find its subgroups. We use the grids to color the kaleidocycles because we can easily see where each face can go when we observe a colored grid instead of the colored object. Using the grid in Figure 3.2 we must first understand that face 1 can only move to odd-numbered faces. Consequently, even-numbered faces can only move to even-numbered faces. First, we can color the three same faces of each tetrahedron in the kaleidocycle to form a subgroup of order 16; see Figure 3.6. Understand that if we constructed this kaleidocycle, the faces that have lines would be fully colored, but for our purposes, we do not color the entire face because we need to see the grid lines.

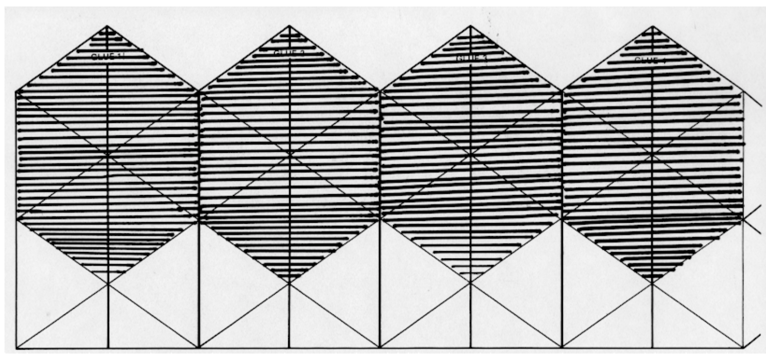

Figure 3.6.

In the grid of Figure 3.6, the triangles in the middle row of triangles can not move to the bottom row of triangles. So, we need to know how many other triangles one of the triangles in the top or third row can move to without moving the triangles in the middle row to the triangles in the bottom row. By observing what the symmetries of this colored kaleidocycle are, we find that the group formed by the symmetries of this object is $D_4 \oplus Z_2$. The group $D_4 \oplus Z_2$ is a subgroup of $D_4 \oplus Z_2 \oplus Z_2$. If we color only two faces of each tetrahedron, as in Figure 3.7, we find a subgroup of order 8 because one pair of colored triangles can only move to another pair of colored triangles.

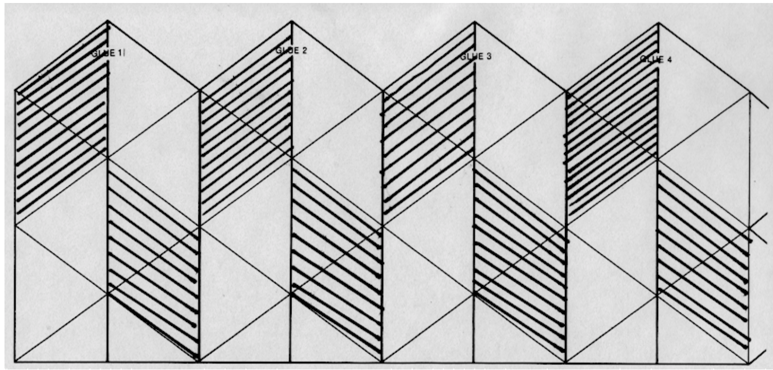


Figure 3.7.

Specifically, we find the subgroup $D_4$. If we color only the same face of every other tetrahedron, as in Figure 3.8, we find a subgroup of order 4 because one colored triangle may only move to another colored triangle.

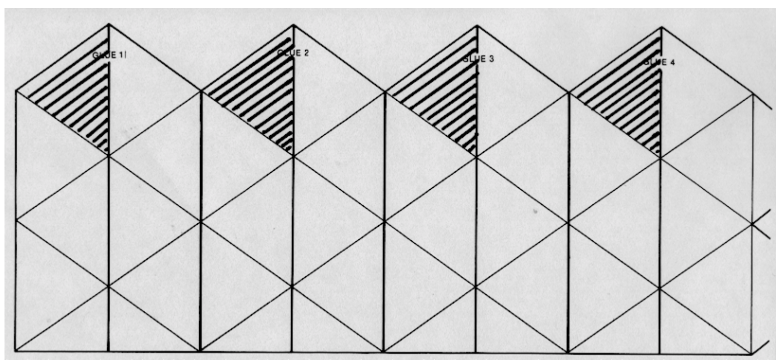


Figure 3.8.

The subgroup of order 4 in this case is the group of rotations of the square kaleidocycle, $Z_4$. By coloring only two faces in the entire square kaleidocycle, specifically, the same face of opposite tetrahedra as in Figure 3.9, we find a subgroup of order 2 because the two colored triangles may switch places.

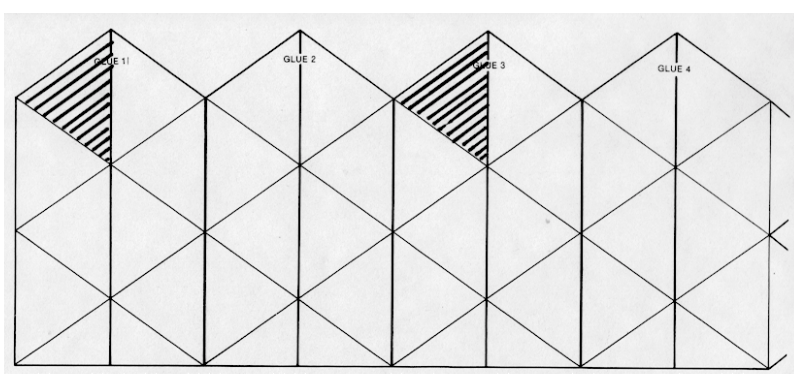


Figure 3.9.

Finally, we can color any one face of any tetrahedron in the kaleidocycle to find the trivial subgroup of the identity, as seen in Figure 3.10.

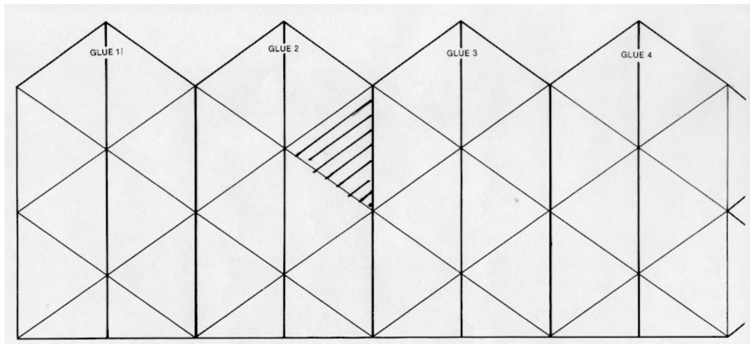


Figure 3.10.

A *triangular kaleidocycle* is a kaleidocycle with six regular tetrahedra. The hexagonal kaleidocycle can be constructed, but it does not have writhing symmetries because no hole exists in the middle of the object and when attempting to writhe the object, the six tetrahedra crash into each other. As seen in Figure 3.11, attaching six regular tetrahedra so that they can writhe will not create a kaleidocycle.

Figure 3.11.

An *n-gonal kaleidocycle* is a kaleidocycle composed of 2*n* regular tetrahedra. Half of the symmetries of the *n*-gonal kaleidocycle include 4*n* elements consisting of *n* elements obtained by rotating the object itself in $360°/n$ increments and the identity, *n* elements obtained by flips along *n* axes through two opposite edges, and 2*n* elements found by composing the rotations and flips with the horizontal reflection of the object. The compositions of the rotations and the flips with the horizontal reflection produce other reflections of the object. We will show, below, that the rotations and the flips develop $D_n$, and the rotations, flips, and the compositions form $D_n \oplus Z_2$.

**Lemma 3.12:** *The group of rotations, flips, and the compositions of these elements with the horizontal reflection of an n-gonal kaleidocycle is isomorphic to* $D_n \oplus Z_2$.

**Proof:** Let *G* be the group of rotations, flips, and the compositions of these elements with the horizontal reflection of an *n*-gonal kaleidocycle. Let the horizontal reflection

$(1,3)(5,7)...(8n-7,8n-5)(8n-3,8n-1)$ be equal to $q$. Define a function $\varphi$ from $G$ to $D_n \oplus Z_2$ where $\alpha$ is mapped to $(\alpha,0)$ and $\alpha(q)$ is mapped to $(\alpha,1)$. To prove that the two groups are isomorphic, we must show $\varphi(\sigma\tau) = \varphi(\sigma)\varphi(\tau)$ for all $\sigma$ and $\tau$ in $G$. First, we must show that $\varphi$ is both one-to-one and onto. Assume $\varphi(\alpha) = \varphi(\beta)$. Then, $(\alpha,0) = (\beta,0)$. Thus, $\alpha = \beta$. If we assume that $\varphi(\alpha(q)) = \varphi(\beta(q))$, then $\alpha = \beta$. However, $\varphi(\alpha) \neq \varphi(\beta(q))$ because $\alpha$ will be mapped to $(\alpha,0)$ and $\beta(15)$ will be mapped to $(\beta,1)$ and $\alpha \neq \beta$. Let $(\delta, k)$ be an element of $D_n \oplus Z_2$. If $k = 0$, then $\delta = \alpha$, and if $k = 1$, then $\delta = \alpha(q)$.

Now, to prove the isomorphism we have three cases:

**Case 1:** $\sigma = \alpha, \tau = \beta$

$\varphi(\sigma\tau) = \varphi(\alpha\beta) = (\alpha\beta,0)$ and $\varphi(\sigma)\varphi(\tau) = \varphi(\alpha)\varphi(\beta) = (\alpha,0)(\beta,0) = (\alpha\beta,0)$

**Case 2:** $\sigma = \alpha(q), \tau = \beta$

$\varphi(\sigma\tau) = \varphi(\alpha(q)\beta) = (\alpha\beta,1)$ and $\varphi(\sigma)\varphi(\tau) = \varphi(\alpha(q))\varphi(\beta) = (\alpha,1)(\beta,0) = (\alpha\beta,1)$

**Case 3:** $\sigma = \alpha(q), \tau = \beta(q)$

$\varphi(\sigma\tau) = \varphi(\alpha(q)\beta(q)) = (\alpha\beta,0)$ by Lemma 3.4, and $\varphi(\sigma)\varphi(\tau) = \varphi(\alpha(q))\varphi(\beta(q))$
$= (\alpha,1)(\beta,1) = (\alpha\beta,0)$

Thus, $G$ is isomorphic to $D_n \oplus Z_2$.■

**Theorem 3.13:** *The group of all of the symmetries of the n-gonal kaleidocycle, denoted* $Kal_n$ *, is* $D_n \oplus Z_2 \oplus Z_2$.

**Proof:** As shown in Lemma 3.12, the group of rotations, flips, and the compositions of these elements with the horizontal reflection of an $n$-gonal kaleidocycle is isomorphic to $D_n \oplus Z_2$. Then, composing these $4n$ elements with the writhe, denoted here by $w = (1,3)(2,4)(5,7)(6,8)...(8n-6,8n-4)(8n-3,8n-1)(8n-2,8n)$. Composing the rotations, flips, and the compositions of rotations and flips with the writhe produces another $4n$ elements, or a copy of $D_n \oplus Z_2$. Thus, $Kal_n \approx D_n \oplus Z_2 \oplus Z_2$ and can be written as

$Kal_n = \langle r, f, F, w \mid r^n = f^2 = F^2 = w^2 = e, frf = r^{n-1}, fw = wf, rw = wr, Fr = rF, Ff = fF, wF = Fw \rangle$

where $r$ is one rotation of the object $360°/n$, $f$ is the vertical flip, $F$ is the horizontal reflection, and $w$ is the writhe.■

One interesting note is that if we put a half-twist in a string of regular tetrahedra before attaching the two ends of the string to make the string of tetrahedra into a Möbius strip, we find some unusual symmetries. Besides the identity, the Möbius strip can be rotated in units of 2 tetrahedra along the center line of the Möbius strip. Thus, the symmetry group of the kaleidocycle with a Möbius twist is isomorphic to $Z_n$.

Now, we move from viewing tetrahedra in various ways to viewing cubes in various ways.

## 4. Cubes

A *cube* is a regular solid with six congruent square faces as seen in Figure 4.1.

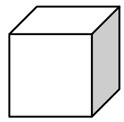

Figure 4.1

The cube has many interesting symmetries. Three axes of rotation of the cube are through the centers of two opposite squares (each axis yields three rotations of 90° increments). Four other axes of rotation of the cube are through two antipodal vertices (each axis produces two rotations of 120° increments), and opposite edges provide another six axes of rotation (each axis bears one rotation of 180°). These 13 axes of rotation along with the identity produce 24 symmetries of the cube. The group of rotations of the cube form $S_4$. Composing these 24 elements with the element $(1,1')(2,2')(3,3')(4,4')$ where 1' is the antipodal vertex of 1 produces another copy of the group $S_4$ for a total of 48 symmetries of the cube. Thus, we show that the symmetry group of the cube is $S_4 \oplus Z_2$.

**Lemma 4.2:** *The group of symmetries of the cube is isomorphic to* $S_4 \oplus Z_2$.

**Proof:** Let *G* be the group of symmetries of the cube. Let $q = (1,1')(2,2')(3,3')(4,4')$, then *q* is in *G*. From here, the proof is the same as the proof of Lemma 3.12. Thus, *G* is isomorphic to $S_4 \oplus Z_2$.■

As with tetrahedra, we want to explore the symmetry groups of two cubes arranged in various formats before we explore the symmetry groups of cubes arranged in strings like kaleidocycles. We will begin with two cubes attached on one face; see Figure 4.3.

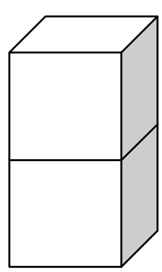

Figure 4.3.

Along with the identity, the object has two types of axes of rotation: one through the centers of the two faces farthest from each other (the axis produces three rotations of 90° increments), two through the midpoints of opposite edges of the face at which the cubes are attached (each axis produces one 180° rotation), and two through opposite vertices of the face at which the cubes are attached (which yields one 180° rotation). Besides the eight rotational symmetries of the object, it has five planes of reflection. One plane of reflection contains the face at which the two cubes are attached. Two planes of reflection contain three diagonals of the horizontal faces. The object also has two perpendicular planes of reflection that bisects the object vertically. Composing the reflection of the face at which the two cubes are attached and the rotations through the centers of the two farthest faces produces three symmetries that are neither rotations or reflections. We will show that two cubes attached on a face has 16 symmetries and forms the group $D_4 \oplus Z_2$.

**Lemma 4.4:** *The group of symmetries of two cubes attached on one face is isomorphic to the group* $D_4 \oplus Z_2$.

**Proof:** The proof is similar to the proof of Lemma 2.3 with $D_4$ instead of $D_3$.■

Attaching two cubes along an edge such that no faces of the cubes touch yields fewer symmetries than two cubes attached at a face, but it will be important as we move into determining the symmetry groups of cubeocycles. This object has two cubes attached along an edge such that the angles between the two cubes are 90°; see Figure 4.5.

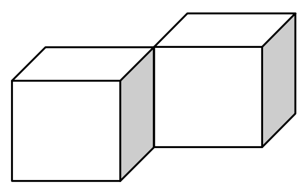

Figure 4.5.

The following four symmetries of this object form the group $K_4$: the identity, one 180° rotation about the axis containing the edge at which the two cubes are attached, denoted by $l$, one 180° rotation about the axis perpendicular to $l$ and touches no other part of the object (this rotation flips the object over so that the two edges farthest apart switch places), and one 180° rotation about the axis that is perpendicular to $l$ and bisects the two edges that are farthest apart. The other symmetries of this object include the reflection across the plane that contains $l$, the reflection across the plane that contains $l$ and the two edges farthest apart, and reflection across the plane that contains the midpoint of $l$. Composing the second plane of reflection with the first axis of rotation provides one additional symmetry that is neither a rotation or a reflection. The three symmetries that are categorized as reflections and the additional symmetry form the group $K_4$. We will show that the symmetry group of two cubes attached at one edge is $K_4 \oplus Z_2$.

**Lemma 4.6:** *The symmetry group of two cubes attached at one edge is isomorphic to* $K_4 \oplus Z_2$.

The natural next step is to explore the ways that three cubes can be attached so that the resultant has symmetry. Three cubes can be attached in a number of ways; some of these ways have symmetries and some of these ways have trivial symmetry groups. Because exploring the symmetry groups of three attached cubes will not assist us in our attempt to characterize the symmetry group of cubeocycles, we will only mention some ways that three cubes can be attached. Three cubes can be attached at faces such that three cubes are on top of each other or at faces such that the object has an L-shape; see Figure 4.7.

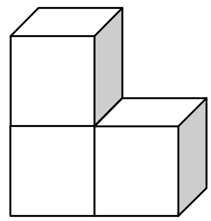

Figure 4.7.

They can also be attached along one edge (see Figure 4.8 as an overhead view of the object), at opposite edges (see Figure 4.9), or along equivalent edges of the cubes such that no faces of the three cubes touch (see Figure 4.10 as an overhead view of the object).

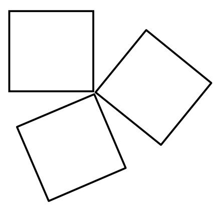

Figure 4.8.

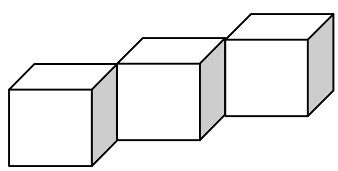

Figure 4.9.

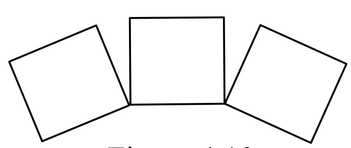

Figure 4.10.

Three cubes can be attached at antipodal vertices to create a short string of cubes (see Figure 4.11) and at one vertex in a number of different ways.

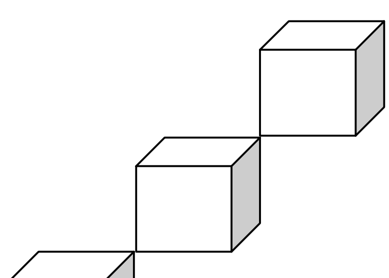

Figure 4.11.

All of these ways to attach three cubes can be altered slightly to create certain symmetries.

Just like with tetrahedra, we need to know if four cubes can be attached in any way to form a cycle. However, unlike tetrahedra, four cubes can be attached along one edge and four faces. This does create a small cycle of cubes, but the cycle does not provide us with writhing symmetries. Thus, our next step is to determine how cubes can be arranged in cycles, and then to discover the symmetry groups of these objects.

## 5. Cubeocycles

Before we define *cubeocyles*, we first need to discuss how cubes can be attached in cycles. Cubes can be attached at opposite edges such that an overhead view of the object would look like Figure 5.1.

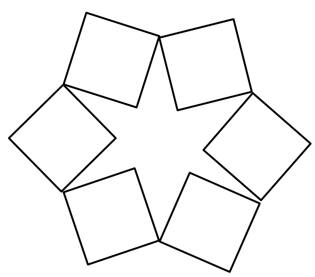

Figure 5.1

This object has a number of symmetries including the identity and five rotations of 60° increments around the axis through the center of the hole of the object. The object also has three rotations of 180° through the axes containing the midpoints of opposite edges and three rotations of 180° about the axes containing the midpoints of the unattached edges of opposite cubes. These 12 symmetries form the group $D_6$; in fact, it is the same group of symmetries as a hexagon. In addition, the object has three planes of reflection containing opposite connected edges and three planes of reflection containing the unattached edges. The object also has a plane of reflection that contains the midpoints of each vertical edge of a cube; we shall call this reflection the horizontal reflection. Composing the horizontal reflection with the rotations of the object produces five additional symmetries. The symmetries provided by reflections and compositions that are neither rotations nor reflections form another copy of $D_6$. Unlike kaleidocycles, this object has no writhing symmetries because of its construction. In order for the object to writhe, the object would have to be able to have the form seen in Figure 5.2.

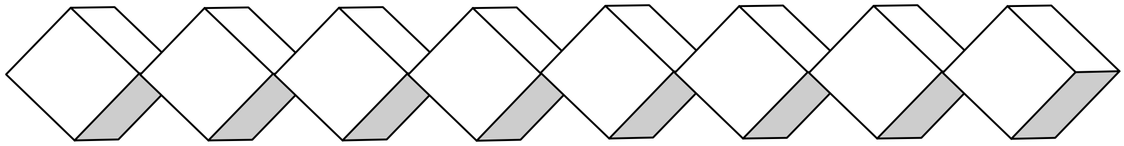

Figure 5.2

Since the object in Figure 5.2 will never be able to be built into a cycle when trying to attach the two outer edges so that the attached edges stay in the horizontal plane, the object in Figure 5.1 has no writhing symmetries. Thus, the symmetry group of this object is $D_6 \oplus Z_2$. We will prove this in the general case.

**Lemma 5.3:** *The group of symmetries of 2n cubes attached on opposite edges such that they are in a cycle is isomorphic to* $D_{2n} \oplus Z_2$.

**Proof:** Let $G$ be the group of symmetries of $2n$ cubes attached on opposite edges. The object will have $2n$ rotations of $360°/2n$ increments. In addition, the object will have $n$ rotations of 180° through the axes containing the midpoints of opposite edges and $n$ rotations of 180° about the axes containing the midpoints of the unattached edges of opposite cubes. This group of $4n$ elements forms the group $D_{2n}$. In addition, the object has $n$ planes of reflection containing opposite connected edges and $n$ planes of reflection containing the unattached edges. The object also has a horizontal reflection. Composing the horizontal reflection with the rotations of the object produces $2n-1$ additional symmetries. The symmetries provided by reflections and compositions that are neither rotations nor reflections form another copy of $D_{2n}$.

All we have to do to show that there are no other symmetries is to determine the number of faces to which one face of a cube can go. If we focus on a top face, call it $y$, it can go to any other face that is on top or on bottom. Thus, there are $2n$ symmetries. In addition, every time $y$ goes to another top or bottom face, there are only two other ways which the other faces can be arranged around $y$. This is true because the outer faces cannot move to the inside faces because the object does not have any writhing symmetries. Also, if one of the inside faces, denote it face 1, moves to another inside face, denote face 2, then the outside face adjacent to face 1 will move to the outside face adjacent to face 2. Thus, the group only has $4n$ elements. Since we have found all $4n$ elements, we know that we have all of the symmetries.

Thus, $G$ isomorphic to $D_{2n} \oplus Z_2$.■

**Definition 5.4:** A *cubeocycle* is a three-dimensional ring made from a chain of an even number of cubes attached at antipodal vertices; see Figure 5.5.

Figure 5.5.

Cubeocycles are another way to attach cubes in a cycle such that the object formed has a number of interesting symmetries. In addition, cubeocycles are similar to kaleidocycles in that they have a writhing symmetry. In our construction, we used wire to thread though the cubes at antipodal vertices to keep the object stable. Our first goal in attempting to find the symmetry group is to discover the number of places that each face can go and how many different ways the other faces can be arranged around that face. Unlike kaleidocycles, one specified face can go to any other face in two different ways; thus, the symmetry group must correspond to a larger group than that of the kaleidocycle.

The cubeocycle with eight cubes has the identity and three rotations of 90º increments as well as four rotations of 180º about the axes containing opposite vertices. Then, composing these eight elements with the horizontal reflection produces another eight elements for a total of 16 elements. Composing these 16 elements with the four reflections through opposite vertices at which cubes are attached yields another 16 elements. Finally, composing these 32 elements with the two writhes produces 64 symmetries; thus, the group has 96 symmetries.

**Theorem 5.6:** *Some of the symmetries of a cubeocycle with 2n cubes is isomorphic to the group* $D_n \oplus Z_2$.

**Proof:** Let *G* be the group of symmetries of the cubeocycle with 2*n* cubes. The cubeocycle will have the identity and $n-1$ rotations of $360°/2n$ increments about the axis through the center of the hole of the cubeocycle (denote these rotations as *r*) and *n* axes of rotation through opposite vertices at which cubes are attached where each axis yields a rotation of 180º (denote these rotations as *f*). These 2*n* elements compose the group $D_n$. Then, composing these 2*n* elements with the reflection through the vertical plane of reflection through two opposite vertices at which cubes are attached provides another copy of $D_n$, which forms the group $D_n \oplus Z_2$.■

Notice that the object also has a horizontal reflection (denote it as *F*) as well as two writhing symmetries (denote it as *w*) that will be discussed in the next lemma.

**Lemma 5.7:** *If F and w are the horizontal reflection and writhe of a cubeocycle, then* $FwF = w^2$ *and the group generated by F and w is isomorphic to the group* $P = \langle w, F | w^3 = F^2 = e, FwF = w^2 \rangle$.

**Proof:** Let the numbers on the faces of the cube in Figure 5.8 correspond to the numbers we will use in the permutations of this proof.

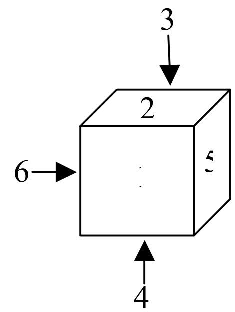


Figure 5.8.

We can isolate our attention to just one cube since *F* and *w* do not permute the cubes and they just rearrange the faces of the cubes. Then, $w = (145)(263)$ and $F = (15)(63)$. Then, $FwF = (15)(63)(145)(263)(15)(63) = (154)(236)$. Since $w^2 = (145)(263)(145)(263) = (154)(236)$, $FwF = (154)(236) = w^2$. The group generated by *F* and *w* is $P = \langle w, F | w^3 = F^2 = e, FwF = w^2 \rangle$. Obviously, *P* is isomorphic to $D_3$.■

**Corollary 5.9:** *The group of all of the symmetries of a cubeocycle with 2n cubes, denoted* $Cub_n$, *is isomorphic to the group* $D_n \oplus Z_2 \oplus D_3$.

**Proof:** Let *G* be the group of symmetries of a cubeocycle with 2*n* cubes. By Theorem 5.6, we know that the rotations and the compositions of the vertical reflection form a group that is isomorphic to $D_n \oplus Z_2$. Composing any of the elements in the group $D_n \oplus Z_2$ with any product of $F^j w^k$ where *j* can equal 0 or 1 and *k* can equal 0, 1, or 2 (where *F* and *w* come from Lemma 5.7) produces 24*n* elements, which forms a group that is isomorphic to $D_n \oplus Z_2 \oplus D_3$ that has 24*n* elements.

Now we must show that there are no other symmetries of a cubeocycle. Similar to Lemma 5.3, we need to determine the number of faces to which one face can go. If we focus on one face, call it *y*, we need to know the number of other faces to which it can move. First, *y* can go to every other face once, so since there are 2*n* cubes and each cube has 6 faces, *y* can move to 12*n* other faces. Then, the other faces around *y* can be arranged in only 2 ways. This is true because the six faces that come together at an attached vertex must come together in the same way since the two cubes are attached in a certain manner. For example, if *y* is fixed and the two adjacent faces, call them *x* and *z*, that are attached to the same vertex (that attaches two cubes) as *y*, then *x* and *z* can be arranged in two different ways around *y*. In addition, the faces that are opposite to *y*, *x*, and *z* must follow the same movements on their cube as *y*, *x*, and *z* do on the cube to which they belong. Thus, there are no other symmetries of a cubeocycle besides the 24*n* that were already described.

Thus, *G* is isomorphic to $D_n \oplus Z_2 \oplus D_3$.■

Thus, $Cub_n \approx D_n \oplus Z_2 \oplus D_3$ and can be written as

$$Cub_n = \left\langle r, f, F, w \middle| r^n = f^2 = F^2 = w^2 = e, frf = r^{n-1}, Fr = rF, wr = rw, fF = Ff, FwF = w^2 \right\rangle.$$

## 6. Future Research

We have discovered the symmetry groups of a number of objects including kaleidocycles and cubeocycles. Furthermore, we also know the symmetry groups of octahedrons attached at antipodal vertices such that they are in a cycle because octahedrons are the duals of cubes. Therefore, the symmetry group of a cycle composed of dodecahedrons is still needed. If one discovers the symmetry group of dodecahedrons in a cycle, then one will also know the symmetry group of icosahedrons attached in a cycle because a dodecahedron is the dual of an isocahedron.

Another interesting question to investigate is the symmetry group of a cycle formed by attaching an alternating chain of tetrahedra and cubes. After discovering the symmetry groups of kaleidocycles and cubeocycles, a conjecture would be that the symmetry group of the alternating cycle would be a subgroup of the symmetry group of the kaleidocycles and cubeocycles. Moreover, other solids, both regular and non-regular, can be attached in a cycle and one can find the symmetry groups of the objects formed. While there will be fewer symmetries of a cycle composed of non-regular objects, there is the possibility that one can discover an interesting symmetry group of these objects.

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**Appendix A:** Half of the elements of $Kal_4$. The numbers come from the grid of Figure 3.2.

(1)

(1,9,17,25)(2,10,18,26)(3,11,19,27)(4,12,20,28)(5,13,21,29)(6,14,22,30)(7,15,23,31)(8,16,24,32)

(1,17)(2,18)(3,19)(4,20)(5,21)(6,22)(7,23)(8,24)(9,25)(10,26)(11,27)(12,28)(13,29)(14,30)(15,31)(16,32)

(1,25,17,9)(2,26,18,10)(3,27,19,11)(4,28,20,12)(5,29,21,13)(6,30,22,14)(7,31,23,15)(8,32,24,16)

(1,31)(2,30)(3,29)(4,32)(5,27)(6,26)(7,25)(8,28)(9,23)(10,22)(11,21)(12,24)(13,19)(14,18)(15,17)(16,20)

(1,15)(2,14)(3,13)(4,16)(5,11)(6,10)(7,9)(8,12)(17,31)(18,30)(19,29)(20,32)(21,27)(22,26)(23,25)(24,28)

(1,23)(2,22)(3,21)(4,24)(5,19)(6,18)(7,17)(8,20)(9,15)(10,14)(11,13)(12,16)(25,31)(26,30)(27,29)(28,32)

(1,7)(2,6)(3,5)(4,8)(9,31)(10,30)(11,29)(12,32)(13,27)(14,26)(15,25)(16,28)(17,23)(18,22)(19,21)(20,24)

(1, 3) (5, 7) (9, 11) (13, 15) (17, 19) (21, 23) (25, 27) (29, 31)

(1,11,17,27)(2,10,18,26)(3,9,19,25)(4,12,20,28)(5,15,21,31)(6,14,22,30)(7,13,23,29)(8,16,24,32)

(1,19)(2,18)(3,17)(4,20)(5,23)(6,22)(7,21)(8,24)(9,27)(10,26)(11,25)(12,28)(13,31)(14,30)(15,29)(16,32)

(1,27,17,11)(2,26,18,10)(3,25,19,9)(4,28,20,12)(5,31,21,15)(6,30,22,14)(7,29,23,13)(8,32,24,16)

(1,29)(2,30)(3,31)(4,32)(5,25)(6,26)(7,27)(8,28)(9,21)(10,22)(11,23)(12,24)(13,17)(14,18)(15,19)(16,20)

(1,13)(2,14)(3,15)(4,16)(5,9)(6,10)(7,11)(8,12)(17,29)(18,30)(19,31)(20,32)(21,25)(22,26)(23,27)(24,28)

(1,21)(2,22)(3,23)(4,24)(5,17)(6,18)(7,19)(8,20)(9,13)(10,14)(11,15)(12,16)(25,29)(26,30)(27,31)(28,32)

(1,5)(2,6)(3,7)(4,8)(9,29)(10,30)(11,31)(12,32)(13,25)(14,26)(15,27)(16,28)(17,21)(18,22)(19,23)(20,24)