ICOSAHEDRON AND Φ

Although mathematicians tend to order and axiomatize our expositions, in this article also we'll try to keep us faithful to the heuristic of our particular process.

I. - *A* curious construction.

a) If we cut two same golden rectangles perpendicularly and opposing dimensions

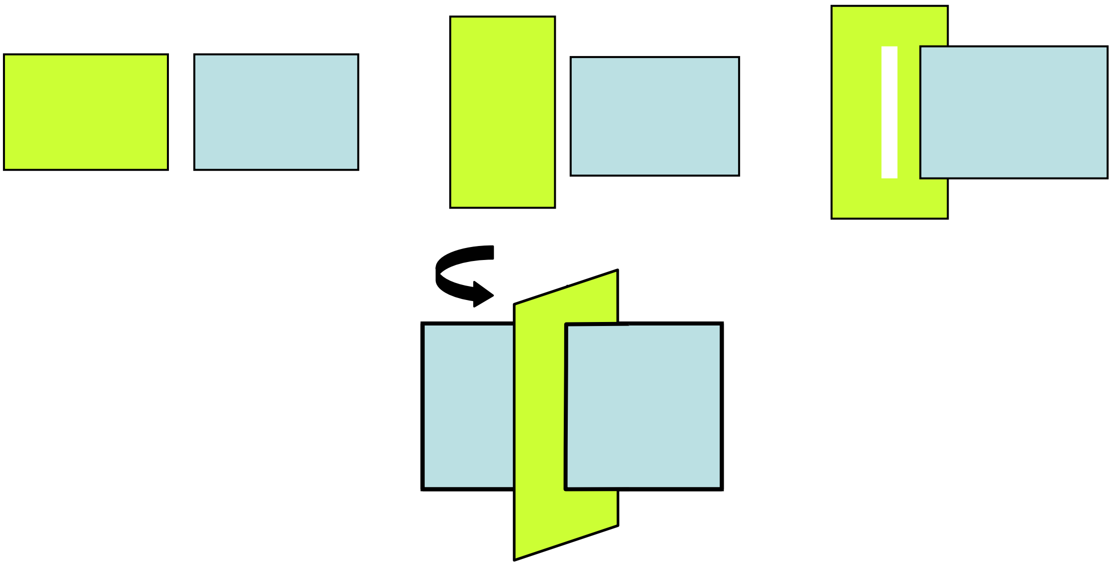


Fig1

It is easy to see from Figure 2 that when we join each vertex of one rectangle with the two nearest ones of the other rectangle, we obtain eight new segments of similar length **a**, being **a** the shortest dimension in the golden rectangles;

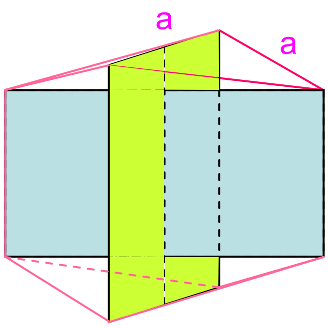


Fig2

b) If we repeat the process with a third golden rectangle, which equals the previous ones, we will note that this new rectangle intersects the other ones in the same way:

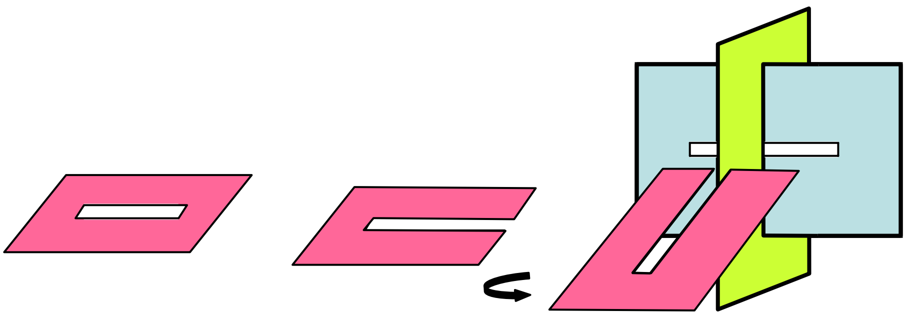


Fig3

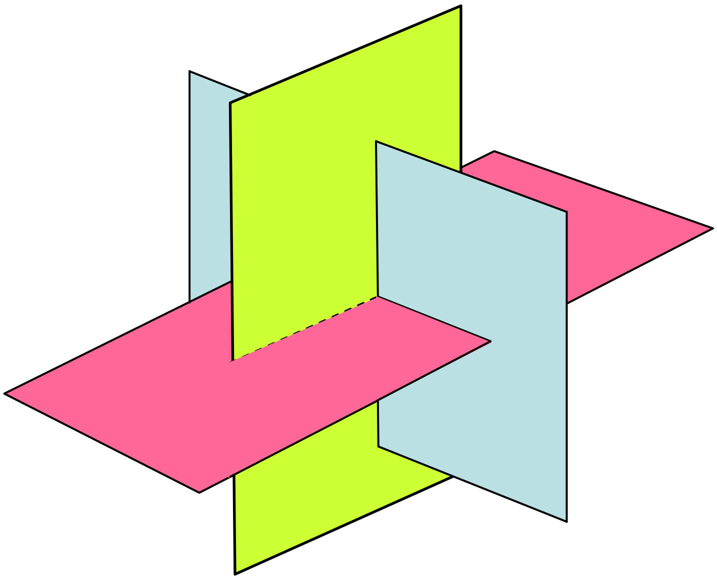


Fig4

Eight segments are now obtained from the intersection of the third and the first rectangle and another eight segments from the third and the second ones:

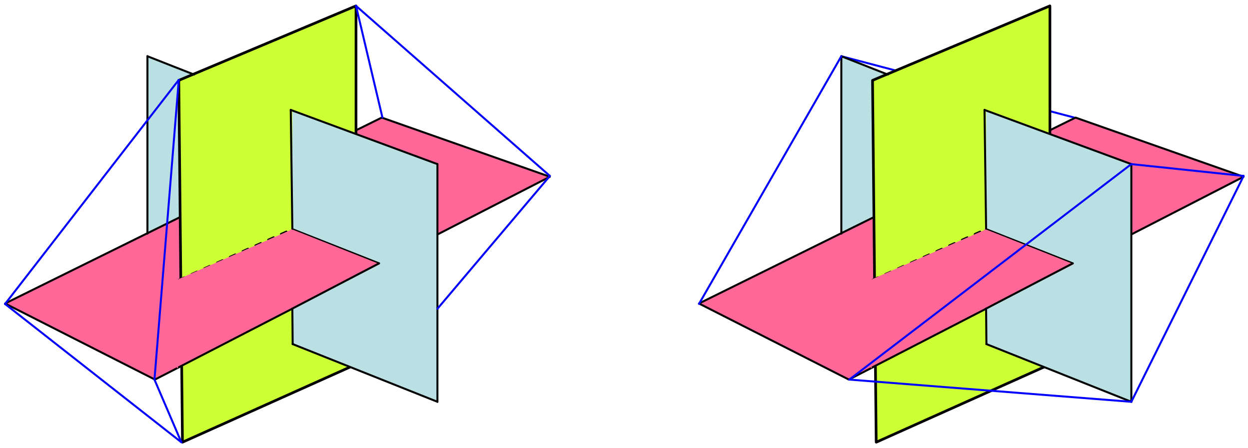


Fig5

That is, we have 8x3 = 24 new edges with length **a** that added to the 2x3 = 6 edges from the three rectangles with length **a** too, give us the 30 identical edges that form 20 equilateral triangles and it is obvious that 12 (4x3) vertexes are only involved (it is easily observable that 5 edges converge in each vertex and only one is a side of one of the rectangles).

[Name Greek. **Eikosaedron** : **eikosi** twenty + **hedra** seat, base.]

From the reading of Divine Proportion [Luca Pacioli 1509] we deduce that some mathematicians of that time knew the golden relationships in the icosahedron thoroughly. Some of these relationships are the base of the previous construction.

II. -Proof

Let a golden rectangle with sides **a** and **aΦ**

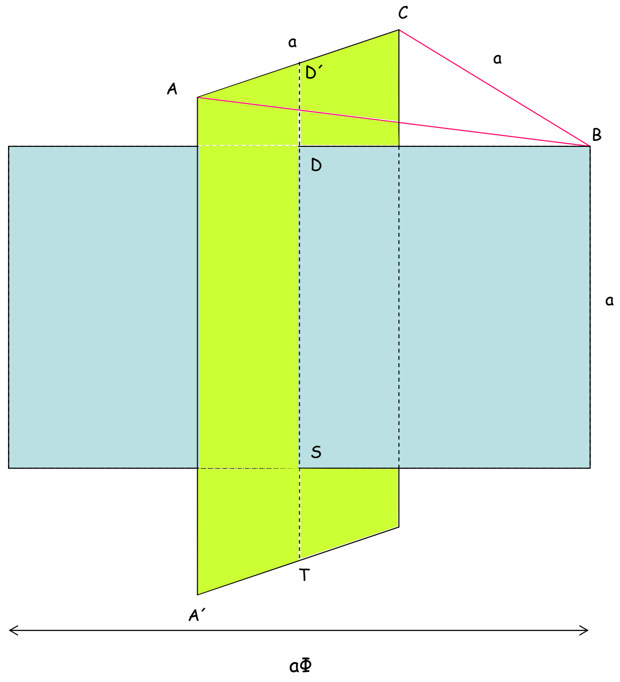


Fig 6

$$\overline{DD'} = \frac{\overline{D'T} - \overline{DS}}{2} = \frac{\Phi a - a}{2}$$

$$\overline{D'B}^2 = \overline{D'D}^2 + \overline{DB}^2 = \left(\frac{\Phi a - a}{2}\right)^2 + \left(\frac{a\Phi}{2}\right)^2 = \frac{2a^2\Phi^2 - 2a^2\Phi + a^2}{4} = \frac{a^2}{4}(2\Phi + 2 - 2\Phi + 1) = \frac{3}{4}a^2$$

$$\overline{AB}^2 = \overline{AD'}^2 + \overline{D'B}^2 = \frac{a^2}{4} + \frac{3a^2}{4} = a^2 \quad \Rightarrow \quad \overline{AB} = a$$

III. – Let's analyse the previous process

Let an icosahedron with side **a**

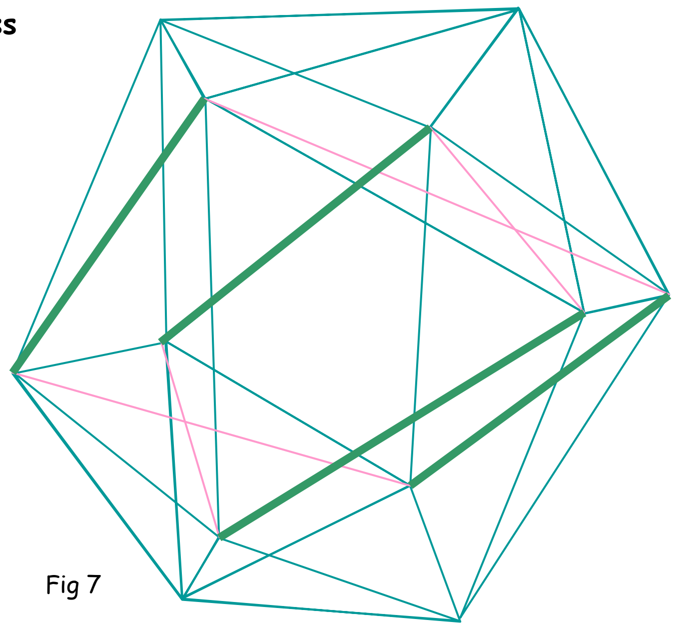


Fig 7

Let's face opposed edges. We obtain several rectangles whose shortest sides are the icosahedron sides of length **a**, and whose longest sides are the diagonals of the regular pentagons formed with the five faces of the icosahedron that converge in the same vertex and, therefore, their length is **aΦ** as we will prove at the end of this section (the result is a classic in Mathematics). So, they are golden rectangles.
The icosahedron has 30 edges and if we face them of two by two as we have just made, we will obtain 15 identical golden rectangles.
Therefore, *whenever we take three any of these rectangles, on condition that two of the shortest sides (edges) don't converge on the same vertex of the icosahedron, they will be able to generate the icosahedron itself.*

There are only two possibilities:

CASE 1: The three rectangles are perpendicular, as in the first construction.

CASE 2: The three rectangles intersect making dihedral angles of 60° in the following way:

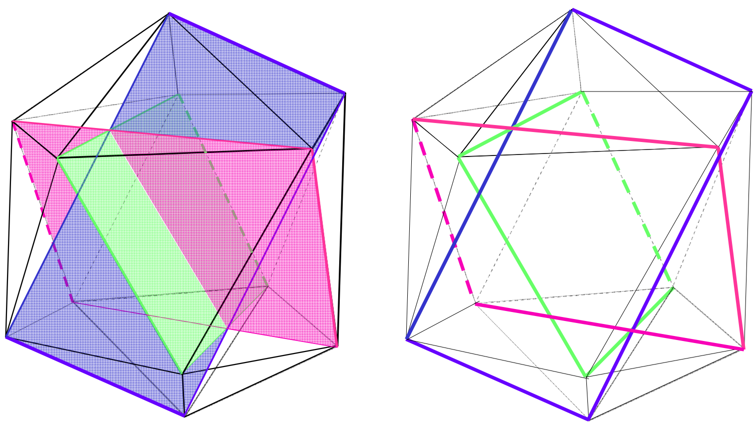


Fig 8

We need a series of results to justify this statement.

Pentagon and Φ

The icosahedron structure is closely connected with the pentagon structure, since on both we can observe the same relationships.

Let us consider a regular pentagon with side **a** circumscribed by a regular decagon with side *x*.

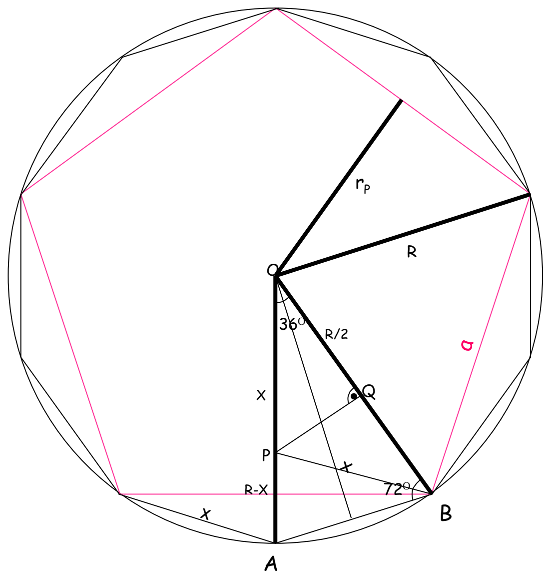


Fig 9

Let R the radius of the circumference that circumscribes both regular polygons.

The triangle OAB is isosceles and its angles are: 36°, 72° and 72°.

When we trace the bisector on the angle OBA, we obtain a triangle PAB that is also isosceles with angles: 36°, 72° and 72°. Therefore, $\overline{PB} = \overline{AB} = x$

The angles in the triangle OPB are: 36°, 36° and 180° - 2.36° = 108°, so it's also isosceles, and therefore $\overline{OP} = \overline{PB} = \overline{AB} = x$ and $\overline{PA} = R - x$.

If we remember that the triangles OAB and BPA are similar:

$$\frac{\overline{OB}}{\overline{PB}} = \frac{\overline{AB}}{\overline{PA}} \Rightarrow \frac{R}{x} = \frac{x}{R-x} \Rightarrow R^2 - Rx = x^2 \Rightarrow$$

$$x^2 + Rx - R^2 = 0 \Rightarrow x = R\left(\frac{-1+\sqrt{5}}{2}\right) \Rightarrow$$

$$\frac{\overline{OB}}{\overline{AB}} = \frac{R}{x} = \frac{R}{\frac{R}{2}\left(-1+\sqrt{5}\right)} = \frac{2}{\sqrt{5}-1} = \frac{2\left(\sqrt{5}+1\right)}{4} = \Phi \Rightarrow R = \Phi x$$

Since the side of the decagon is $x = L_D \Rightarrow L_D = \frac{R}{\Phi}$

$$\boxed{R = L_D \Phi}$$

We trace the height $\overline{PQ}$ on the isosceles triangle OPB thus $\overline{OQ} = \frac{R}{2}$ and

$$\cos 36^{o} = \frac{\overline{OQ}}{\overline{OP}} = \frac{\frac{R}{2}}{\frac{R}{\Phi}} = \frac{\Phi}{2} \qquad \boxed{\cos\frac{\pi}{5} = \frac{\Phi}{2}}$$

If the side of the pentagon is $L_p = a$, then:

$$\sin 36^{o} = \frac{\frac{a}{2}}{R}; \qquad \frac{a}{2} = R\sin 36^{o} \qquad a = 2R\sin 36^{o}$$

$$a = 2R\sqrt{1-\cos^2 36^{o}} = 2R\sqrt{1-\frac{\Phi^2}{4}} = 2R\sqrt{\frac{4-\Phi^2}{4}} = 2R\frac{\sqrt{4-1-\Phi}}{2} \qquad \Rightarrow$$

$$a = R\sqrt{3-\Phi}$$

Notice that $\Phi^2 = \Phi + 1$.

Let's go back to the pentagon again:

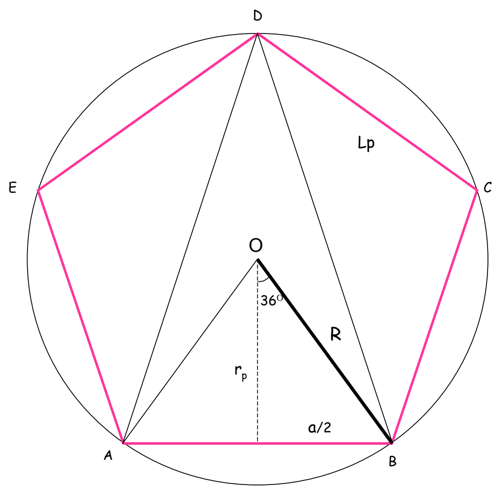


Fig 10

The triangle OAB is isosceles and whose angle O is equals $\frac{360^\circ}{5} = 72^\circ$, so the inside angles in the pentagon are 108°.

If we trace the diagonal $\overline{DB}$ on the pentagon, we'll obtain the isosceles triangle DBC, whose angles are 108°, 36° and 36°.

Let's take now two contiguous diagonals, they make an angle of 36°, and they form part of the isosceles triangle DAB, so that:

$$\overline{AB}^2 = \overline{DA}^2 + \overline{DB}^2 - 2\overline{DA}\cdot\overline{DB}\cdot\cos 36^\circ = 2\cdot\overline{DA}^2 - 2.\overline{DA}^2\cdot\frac{\Phi}{2} = \overline{DA}^2(2-\Phi)$$

$$\overline{DA}^2 = \frac{\overline{AB}^2}{2-\Phi} = \frac{\overline{AB}^2\cdot\Phi^2}{(2-\Phi)\Phi^2} = \frac{\overline{AB}^2\Phi^2}{(2-\Phi)(1+\Phi)} = \frac{\overline{AB}^2\Phi^2}{2+\Phi-\Phi^2} = \frac{\overline{AB}^2\Phi^2}{2+\Phi-1-\Phi} = \overline{AB}^2\Phi^2$$

Therefore: $\overline{DA} = \overline{AB}\Phi$

That is, the diagonal on the pentagon is the side multiplied by Φ

CASE 1: The three rectangles are perpendicular.

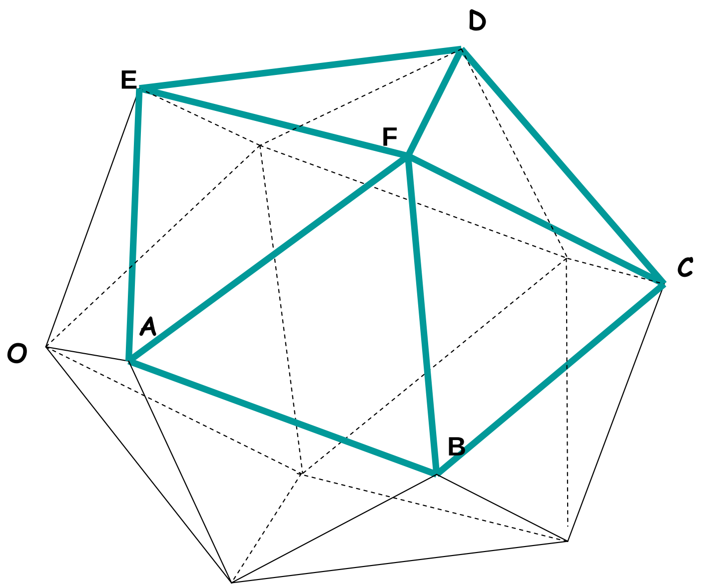


Fig 11

For example, let's take the golden rectangle whose shortest side is AB in the pentagonal pyramid ABCDEF. In this pyramid, the perpendicular rectangle to that one is the rectangle whose shortest side is FD. The third perpendicular rectangle to these two ones could only be the one whose shortest side is OE.
If we start from any other of the five remaining sides of the pentagon ABCDE instead of side AB, we'll obtain another trio of perpendicular golden rectangles. Therefore there are five trios of perpendicular rectangles that will be able to generate the icosahedron; but they are the same in essence.

CASE 2: The three rectangles intersect making dihedral angles of 60º.

In the Figure 11 previous, we started from the golden rectangle whose shorter side is AB in the pentagonal pyramid ABCDEF. The next golden rectangle whose shortest side doesn't leave from the same vertex that the previous one and not perpendicular to it may be the one whose shortest side is EF. In this case, the only third rectangle that satisfies the requirements is that whose shortest side is AD.
If we start from any other of the five remaining sides of the pentagon ABCDE instead of side AB and we proceed in de same way, we'll obtain another trio of perpendicular golden rectangles intersected in angles of 60º that will generate the icosahedron itself.
Those five trios can be obtained by making consecutive clockwise through 72º about the diagonal of the icosahedron whose vertex is F. But, if we make the trio AB, EF, AD turn 72º clockwise about the diagonal of the icosahedron whose vertex is A (origin of the segment AB), we'll obtain another different trio. Repeating that process with each one of the five previous trios we obtain a different trio for each of the first ones. So, they are ten different trios that intersect in angles of 60º (or 120º). But in essence they are identical ones. Obviously if the turn is anticlockwise, the result will be the same.

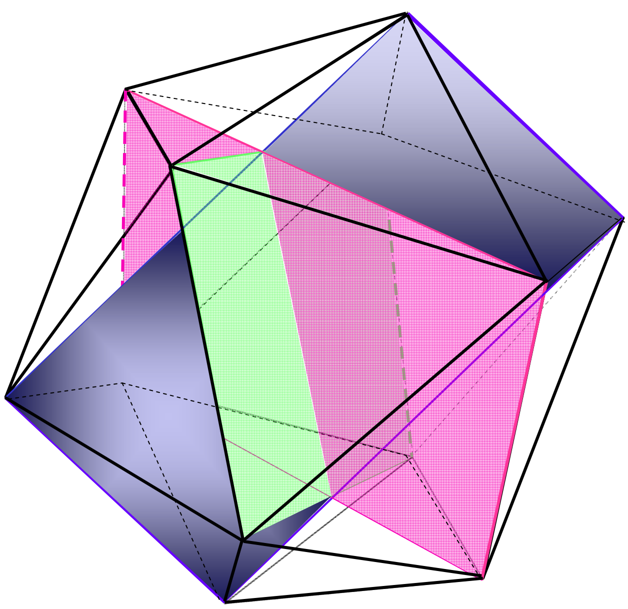


Fig 12

LET'S TAKE ANOTHER WAY…

As we have seen, each vertex of the icosahedron sustains a pentagonal pyramid

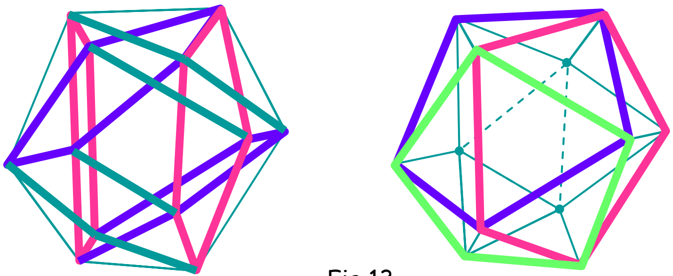


Fig 13

and two vertexes *nor contiguous neither opposed*, are the vertex of the regular pentagon of side **a** as well. Then, the line that joins them is always a diagonal of this pentagon.

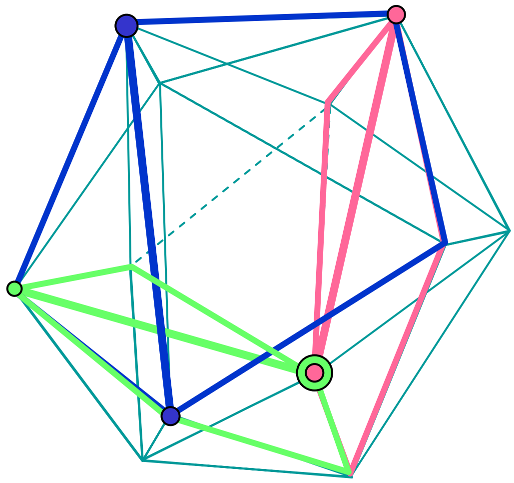


Fig 14

If we trace **all** the diagonals in each pentagon we'll find a new one whose side is $\frac{a}{\Phi^2}$, as we'll see below. So they are 12 new pentagons since there are 12 different pentagonal pyramids.

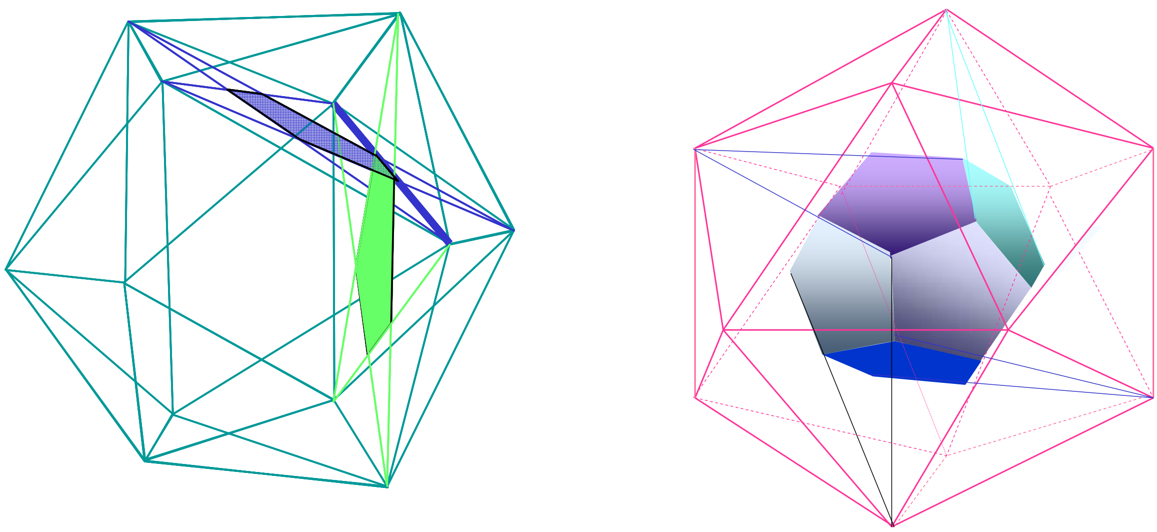


Fig 15

Since each diagonal of the pentagons is common to two pentagonal pyramids that are sustained on two contiguous vertexes (the edge of the icosahedron that unites them is perpendicular to this diagonal) it follows that the twelve pentagons are the faces of a dodecahedron whose side is $\frac{a}{\Phi^2}$.

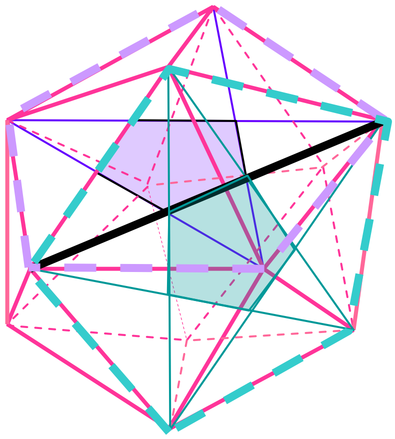


Fig 16

So the 5x 12 = 60 diagonals on the pentagons intersect 3 by 3 to produce the 60: 3 = 20 vertexes of the dodecahedron.

The intersection of these diagonals is interesting. Two of them are always in de same plane and are no concurrent in a vertex of the pentagon (although they clearly intersect at one vertex of the dodecahedron). The third one joins *the spinode of the pyramid whose base is the pentagon which is determined by the two previous diagonals, with the spinode of the contiguous pyramid which has the edge determined by both diagonals in common to the previous pyramid.*

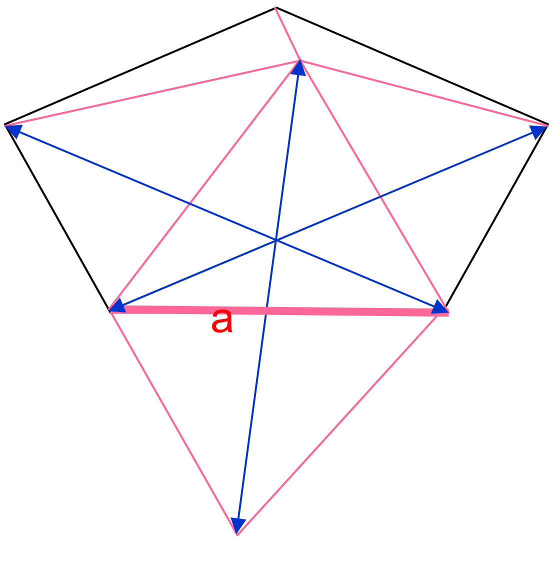


Fig 17

The above is another way to see the trios of no orthogonal golden rectangles. Two of the rectangles have their side **aΦ** on two diagonals (not concurrent in a vertex) of the same pentagon. The third rectangle has it side **aΦ** on the diagonal that joins *the spinode of the pyramid whose base is the pentagon which is determined by the two previous diagonals, with the spinode of the contiguous pyramid which has the edge determined by both diagonals in common to the previous pyramid.*

In the drawing, apart from the construction, you can see that three rectangles of this type determine the 12 vertexes of an icosahedron. The view of the polyhedron is that in which two pentagons are opposed in planes that are parallel to the horizontal plane.

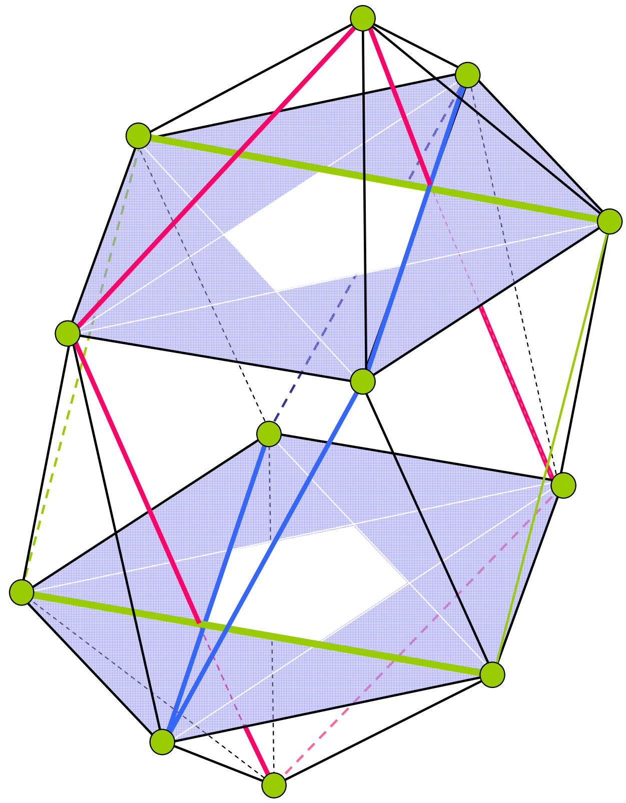
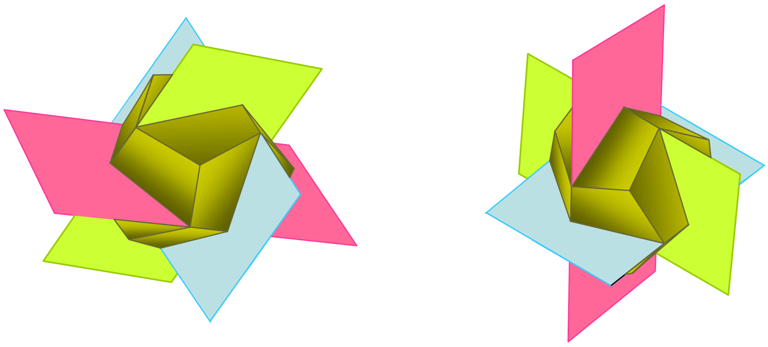


Fig 18

As we have said before, there are 15 different golden rectangles in an icosahedron sine we face its 30 opposed edges two by two. If we intersect three of them in an orthogonal way we have 5 different constructions (starting each from the five edges that converge in a point) that use the 5x3 = 15 aforementioned rectangles.

On the other hand if the construction is **non orthogonal**, there are 10 different ways to choose the trios of rectangles since each one of them intersect along a diagonal of the dodecahedron. As the dodecahedron has 10 diagonals, it follows: 10 diagonals multiplied by three rectangles that determine each diagonal = 30 rectangles; but since rectangle shares two different trios (each diagonal on the pentagons of the icosahedron contains two different vertexes of the dodecahedron), then 30 = 2 x 15 (nº of different golden rectangles).

Case 1



Case 2

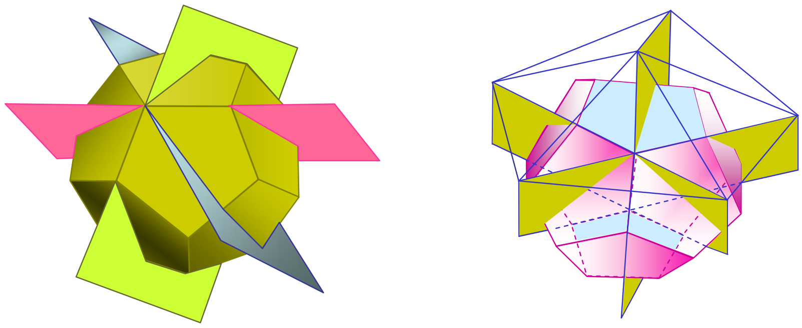


Fig 19

ANALYSING THE ANGLE OF CUT IN CASE 2

We observe from the above that in these trios, two any of the rectangles intersects along the line that joins two opposed vertexes of the dodecahedron (not inscribed).

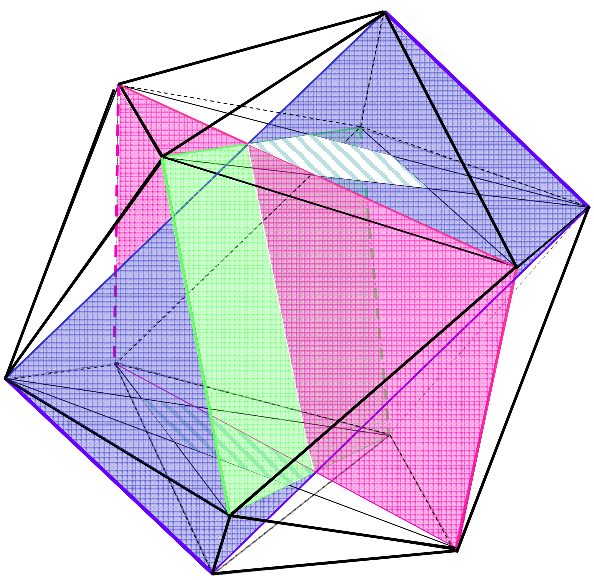


Fig 20

a) Relations between the golden rectangle and this dodecahedron:

A classic construction (pentalfa):

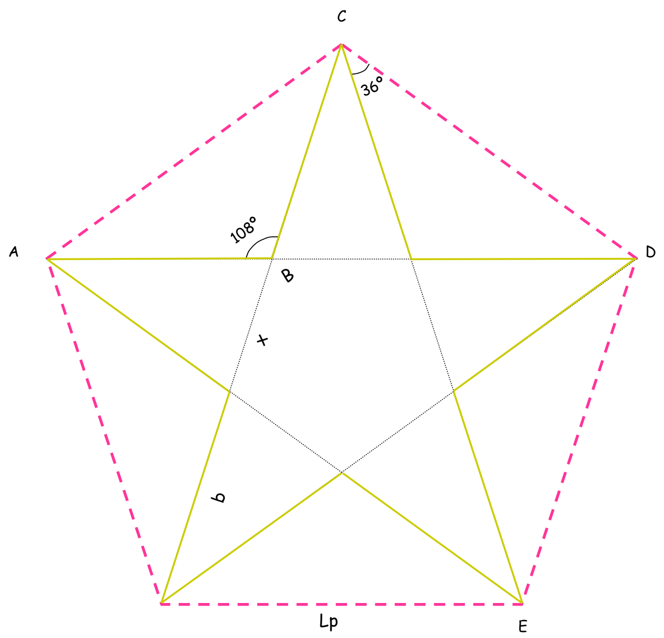


Fig 21

The triangles ABC and CDE are similar since their angles are equal, and

$$\overline{CE} = \Phi L_P \quad \Rightarrow \quad \frac{\Phi L_P}{L_P} = \frac{L_P}{b} \quad \Rightarrow \quad b = \frac{L_P}{\Phi}$$

$$x = \Phi L_P - 2b = \Phi L_P - \frac{2L_P}{\Phi} = \frac{L_P(\Phi^2 - 2)}{\Phi} = \frac{L_P(\Phi - 1)}{\Phi} = L_P\left(1 - \frac{1}{\Phi}\right) = \frac{L_P}{\Phi^2}$$

$$2b + x = 2\frac{L_P}{\Phi} + \frac{L_P}{\Phi^2} = \frac{L_P(2\Phi + 1)}{\Phi^2} = \Phi L_P$$

This measure "x" is the dodecahedron side that is formed inside the icosahedron when you trace every diagonal on the pentagonal bases of the 12 pyramids.

b) Let now two golden rectangles that intersect in a non orthogonal way, that is, along a diagonal of the dodecahedron:

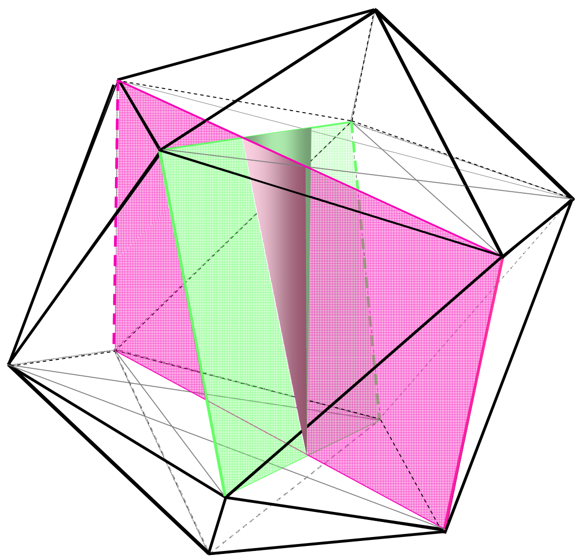


Fig 22

a = side of the icosahedron
a/Φ^2 = side of the pentagon that is face of the dodecahedron

Note that $a/\Phi + a/\Phi^2 + a/\Phi = a\Phi$

It is curious to observe that the two rectangles of measures **a** and **a/Φ** are golden again.

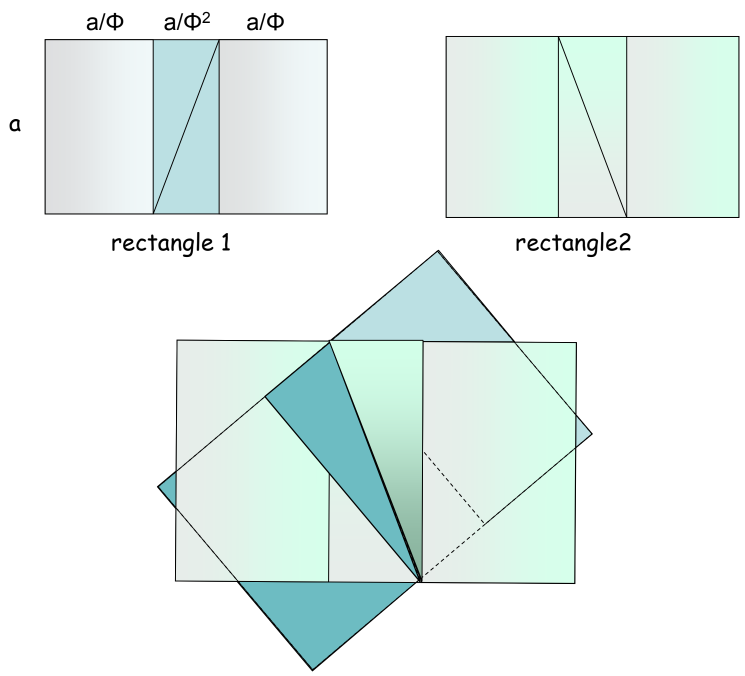


Fig 23

Study of the cut:

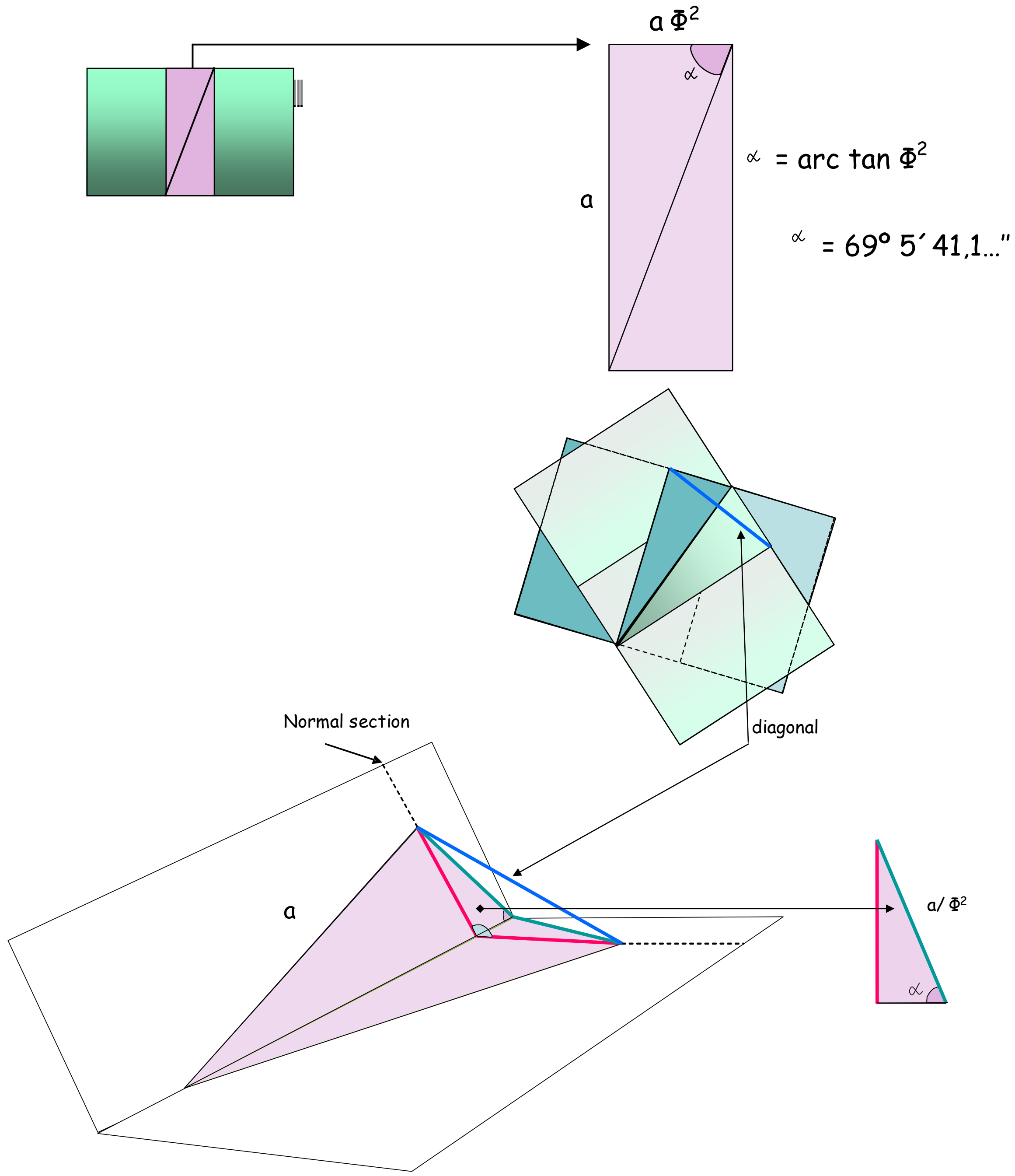


Fig 24

Let α the angle that forms the shortest side with the diagonal in the rectangle **a** x **a/Φ²**, then

$$\sin\,\alpha = \frac{\Phi^2}{\sqrt{1+\Phi^4}}\ ; \quad \cos\,\alpha = \frac{1}{\sqrt{1+\Phi^4}}\ ; \quad \tan\,\alpha = \Phi^2$$

From $\sqrt{\Phi^4+1} = \Phi\sqrt{3}$ is obtained $\sin\,\alpha = \frac{\Phi}{\sqrt{3}}$; $\cos\,\alpha = \frac{1}{\Phi\sqrt{3}}$; $\tan\alpha = \Phi^2$

Since one of the triangles is formed by two contiguous sides of the pentagon and the corresponding diagonal, if the side is a/Φ^2 the diagonal will be a/Φ.

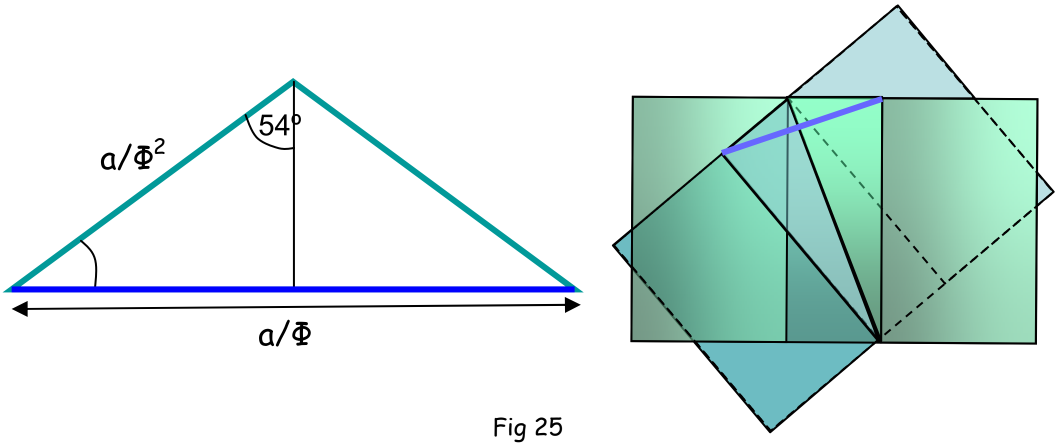


Fig 25

On the other hand the right triangle of the figure 24 is solved:

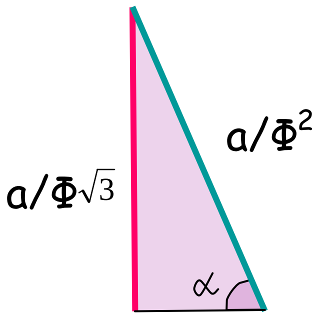


Fig 26

$$\frac{a}{\Phi^2}\sin\alpha = \frac{a}{\Phi\sqrt{3}}$$

So, we only have to considerer an isosceles triangle to calculate the angle of the normal section:

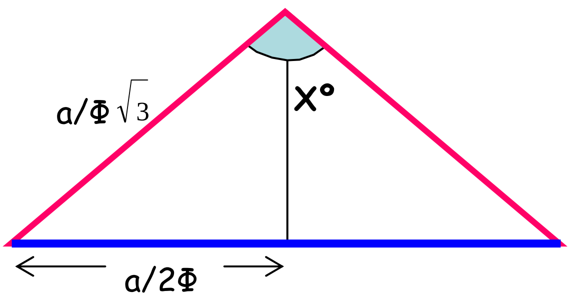


Fig 27

$$sen\frac{x}{2} = \frac{\frac{a}{2\Phi}}{\frac{a}{\Phi\sqrt{3}}} = \frac{\sqrt{3}}{2} \qquad \text{then} \qquad \frac{x}{2} = 60^o \quad y \quad x = 120^o$$

Therefore the cut of both planes is of 120º or 60º

That is, there are only two types of trios of golden rectangles that generate to the icosahedron. The rest of possibilities comes down to one of these.

Acknowledgments: We would like to thank Colleagues of course 99/00 because without them this study had not begun and the professor D. Tomás Recio for its suggestions.

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