It's a long way to the stars

or, The sorry state of polyhedron theory today

Introduction

It's taking me a long time. The stellations of the regular icosahedron caught my imagination in the late 1990s. Stellation theory seemed to have been pretty much wrapped up by then, with the last word being spoken in 1938 when Coxeter, DuVal, Flather and Petrie published *The fifty-nine icosahedra*. But I was dissatisfied, their results didn't seem quite right. I set out to get to the bottom of things and to provide a more useful enumeration – it seemed straightforward enough, a nice little project. But what a journey it has turned out to be! I published a couple of essays and my web pages on stellation and facetting continued to flourish for a while but then went quiet. I am still working on it, and hope eventually to come up with the goods. In the meantime, this note tries to explain what is going on.

Historical overview

The ancient Greeks started it all. By the time of Plato the five convex regular solids had been discovered and Euclid went on to synthesise the geometry of space, constructing Plato's polyhedra as his crowning glory. Star and other non-convex polyhedra become popular during the Renaissance, leading eventually to Kepler's idea of stellation and his discovery of regular stars. A great explosion of synthetic geometry during the 18th and 19th Centuries, mainly in France and Germany, led to Cayley's understanding of the polar or dual relationships between the regular stars, and of their densities. At about the same time, Schläfli discovered higher-dimensional analogues of polygons and polyhedra which Stott later dubbed polytopes; polygons and polyhedra could now be understood as two- and three- dimensional examples of a more general theory of polytopes in any number of dimensions. Along the way various stellations of the regular bodies had been found, but it was not until JCP Miller proposed his set of rules that Coxeter and DuVal could enumerate those of the icosahedron. The final coda seemed to have been written when Bridge, aware of the dual relationship between the processes of stellation and facetting, enumerated the facettings of the regular dodecahedron.

Broken theories

As I traced this process laboriously back through the archives, I became more and more astonished. Only the ideas of non-convexity and higher dimensions stood up to rigorous scrutiny. Every one of the rest is at best half-baked and incomplete, and some are fatally flawed.

Miller's rules turn out to bear little relation to the Keplerian idea of stellation, rather they are based on ideas of spatial decomposition developed by Wheeler and others. They jump into the narrative of *The fifty-nine icosahedra* as if from nowhere. Worse, they deliberately forbid any idea of internal structure and the idea of a "face" was changed subtly in meaning, greatly affecting the result.

Bridge enumerated only a handful of the possible facettings of the dodecahedron, confining himself to what we might call a tidiness of form. He even rejected certain facettings because the dual icosahedra were not tidy enough for him. Many of the fifty-nine icosahedra have duals which were evidently not tidy enough for him either. Even so, in some cases he described the duals of several distinct stellations, with the duals all sharing the same outward form and differing only in their internal structure.

The question, “What is a polyhedron?” has never been properly answered. Definitions have ranged from solids to surfaces to skeleta to combinatorial point sets, to "realisations" of partailly-ordered sets (abstract polytopes) with all sors of features such as infinite extent, coincident elements and so on allowed by some investigators but not by others. Today the debate rages perhaps stronger than ever. Particularly unfortunate is the habit of defining a polyhedron as a solid, then later describing tilings of the plane as "infinite polyhedra", remarking that they differ only in not being solids. This boils down to the logical absurdity that "a tiling is a solid which is not a solid": of course what has actually happened is an unconscious slip from polyhedra as solids to polyhedra as surfaces. Another vexed confusion arises over whether the edges of a polyhedron are necessarily finite segments or may be infinite lines and whether or not it matters.

Our ideas of inside and outside are clouded by the popularity of two different and incompatible approaches; the one based on the idea of "outside" as containing infinitely many straight lines, and the other on the idea of the surface wrapping round a dense interior. The densities determined for regions within self-intersecting polygons and polyhedra depend on which of these approaches we take, whether the surface is orientable or not, how we choose to interpret a density of zero, and even what kind of space we placed our polyhedron in to start with.

The synthesis of geometry leads first not to Euclidean space but to general projective space. This latter space is the home of polarity, and in general the polar or reciprocal of some polyhedron is not at all what we have been led to believe: polyhedral duals are evidently something else.

Bad habits

I am not alone in seeking some way through the wider mess that is polyhedron theory today, and along with other investigators have found various obscuring principles at work.

One class of problem comes from our failure to define the things we are talking about. From the days of Euclid we have consistently failed to define a “polyhedron”. Other ideas, such as their reciprocity or the kind of space we are putting them in, have since fallen into the same trap.

A related class of problem comes from a habit of twisting some well-understood term to a new meaning inconsistent with the old. Miller's rules are a prime example. Another is the way that modern analytical disciplines have adopted various definitions of polyhedra and polytopes which are quite alien to their origins in pure geometry and often inconsistent with each other.

Many principles first enunciated in ages past have become detached from their original context, enshrined in mathematical folklore and applied willy-nilly out of context. The slip from solids to surfaces is one example. Others include; various of those specialised definitions being applied in another discipline, the assumption of Euclidean space, and the idea that every polyhedron has a dual which may be found by reciprocating it about a concentric sphere.

That last is also an example of the more general fault of unjustified generalisation from the particular. Prime examples here include taking results from the study of convex, symmetrical polyhedra and assuming that they apply also to non-convex or asymmetric polyhedra such as star pyramids. For the most part, they do not.

Then there is the habit of concept transfer (noted by Lakatos), a modern example being to provide some rather abstract but at least rigorous algebraic definition, then declaring a geometric polyhedron to be a "realisation" of the abstract form and assuming that the rigour of the abstract concept has solved all the problems. However at best this merely transfers all the old questions onto "what is a realisation?" Worse, it can disguise the fact that some of these abstract forms may not be what we want anyway – the theory of abstract polytopes allows figures which are inconsistent with Euler's and Poincaré's approaches and therefore have no definite surface topology.

And finally there can be a blindness to fundamental mathematical ideas such as continuity (a particular preoccupation of Grünbaum's), or to the distinction between regular polytopes and configurations (seen in that between the quadrilateral polygon and the complete quadrilateral and complete tetragon configurations) and whether it matters.

Stating the problem

The whole thing is utterly shambolic and I am, as I said, astonished that this sorry state of affairs has been allowed to develop. The starting point goes right back to the foundations of geometry itself. As you have just seen, polyhedron theory really is that broken.

Here are nine questions that need answering, with each tending to build on the previous ones. I have now found answers to most of them, which I hope to expand on in due course: a beginning is made in the last section.

1. **What kind, or kinds, of space can we do geometry in?**

   The most basic kinds are incidence spaces, which may be discrete or continuous, finite or infinite. We can do quite a bit of geometry in these, for example we can construct certain partially-ordererd sets called abstract polyhedra (or, more generally, abstract polytopes). Beyond these are what I call morphic spaces, which have a smooth continuity but no idea of absolute measurement, such as projective and affine spaces, and beyond these are concrete metric geometries with measurable lengths and angles, such as Euclidean, spherical, hyperbolic and many others.

2. **Of these, which might be suitable for constructing polyhedra?**

   Incidence spaces are too simple to be able to distinguish polyhedra from other constructions such as configurations. All the other kinds mentioned above are suitable, with projective space being the best starting point.

3. **What exactly is a polyhedron? And for that matter, what is a polygon or any polytope?**

   A polyhedron is basically a lump of stuff whose surface is divided into faces, edges and corner points (vertices). The "stuff" is best thought of as magic rubber - not only can it stretch and shrink but it can pass through itself to allow star surfaces. Crucially, I relax the usual assumption that a polygonal face is necessarily disc-like, for example a star face may be a Möbius band. Similarly, the interior of the polyhedron may be inherently twisted. Indeed, to fully define a given polytope the interior of every element may have an arbitrary (but contiguous) topology and has to be defined. Typically we will want to map a polyhedron into ordinary Euclidean 3-space so that we can see it, although that is not strictly necessary. If our main interest lies with traditional (flat, disc-like faced or epipedal) polyhedra in 3-space, we must be explicit that we are confining ourselves to this variety.

4. **What are the “inside” and “outside” of a polyhedron?**

   The inside is just the lump of rubber (contained within its outer surface). When it is placed in ordinary space, it may be twisted up to create a non-convex figure such as a star. Because we can choose different manifolds (such as a Möbius band) to "fill" inside a given boundary, the resulting star may or may not obey the usual "density" rule, depending on which manifold we choose and how it is then twisted up.

5. **Can a polyhedron have holes that are not topologically toroidal?**

   Yes and no. All faces must be continguous, unbroken surfaces, so such holes are impossible. But some faces can be twisted up to leave toroidal holes in the middle even though the Euler value may disagree. And there can sometimes be "false" holes where the body wraps around some part of empty space, such as a long sausage shape bent round until the ends overlap.

6. **What exactly are polyhedral duality and reciprocation?**

   Duality is a property of the polygonal decomposition of a compact, unbounded surface (it is easily proved using graph theory). Reciprocation is broadly similar to projective polarity, except that the edge polar to a vertex is that segment, of some line, which does not cross infinity (and likewise for faces). Commonly, the polyhedron is reciprocated about a concentric sphere or, if there is no centre of symmetry, the centre of gravity (average position) of its vertices; the result may be called the standard dual of that polyhedron and, if all edges of both polyhedra are tangent to the sphere then it is known as the canonical dual. Duality and reciprocation do not affect the interior characteristics of a polyhedron.

7. **Can we now say that any figure reciprocal to some polyhedron is also a polyhedron?**

   Yes and no. Reciprocation is a purely geometrical operation whose home is in morphic projective space. Here, the answer is yes. But in other spaces the principle of reciprocity is broken and it can result in degenerate figures in which elements of the dual polyhedron may be duplicated or missing and so the answer there is no. Such a degenerate image can sometimes still have a valid "morphic" structure, in which case it is a morphic polyhedron but not a geometrical one.

8. **What exactly are the (reciprocal) processes of stellation and facetting?**

   Stellation is the process of extending the surface elements (sides of a polygon, faces of a polyhedron) until they meet to form a new polyhedron or set of polyhedra. As such, it requires a concrete or metric space such as Euclidean or one of the non-Euclidean geometries. The topology of the stellation may be radically different from the original. It is thus alien to the morphic principles which underlie the previous answers. The essentially metric approach of spatial decomposition into cell sets is even more alien and is likely to prove a minority sport. Meanwhile, the many special cases which arise each needs careful investigation before one can say more.

9. **Which are the usefully distinct stellations and facettings of the regular polyhedra?**

   Morphic theory acknowledges that several distinct star polyhedra may share the same outward form and that elements may overlap or even coincide. This adds significant complexity to any enumeration. My longstanding approach of working with precursors still looks like it is heading in the right direction. One can at least distinguish between a true stellation or facetting which encloses the original core at the centre of a new polyhedron or concentric compound, from a *constellation* in which the polyhedra are discrete. We can also define a *cage*, in which individual polyhedra are also disposed externally to the original core but are also tangent to each other or intersect. Typically, a constellation will be dual to some cage, but the reverse is not always true.

Only the last two questions deal with stellation – the rest are all to do with the general theory of polyhedra. You can perhaps now understand why it is taking me such a long time to get to the bottom of things, especially as I am not in any way a professional mathematician.

A programme of work

Now that I have at last found the starting point, I can begin to build back up towards something that might hold together a little better and perhaps even stand the test of time. From the above analysis, a sequence of distinct stages of work can be developed:

1. Analyse the problem. This essay has pretty much done that.
2. Develop a foundational theory of polyhedra. I call this morphic theory and give a brief pointer below.
3. Clarify polyhedral duality and its relation to polyhedral reciprocation.
4. Understand the dual relationships of stellations and facettings and redefine them accordingly.
5. Apply this understanding to the Platonic solids (the regular tetrahedron, cube, octahedron, icosahedron and dodecahedron.

A morphic theory of polyhedra

I have been developing an approach which I believe can bring all this together into a coherent picture. I call it morphic theory and the rubber objects it deals in morphic polyhedra or, more generally, morphic polytopes. In essence it extends the topological approach by, as mentined, allowing faces or cells which are not simple, while embroidering the abstract approach with a more developed theory of realization. Moreover, it does these in such a way as to bring them together to yield a consistent overall picture.

It has led me to the discovery of a whole new class of regular star polytopes, including various new regular dodecahedra and icosahedra. Many if not all of these regular stars are valid stellations, though universally forbidden by Miller's unfortunate rules. You see, I'm only trying to stellate the icosahedron the way it should be stellated, and look where the journey has been taking me!

In the last few years I have mostly been busy on other things. My investigations into the processes of stellation and facetting are progressing at a snail's pace. At least morphic theory provides me with a starting point, which is more than Kepler and his successors ever had. But where polyhedron theory is grounded in the morphic world of topology, stellation theory is grounded in the metric world of Euclidean space. They sit uneasily together (which is probably why over two thousand years of progress still leave us floundering). This journey is going to be a very different one.

Suggested reading

If you get fed up waiting or prefer to tread the path for yourself, here is some suggested reading, broadly in order of progress from the ground up:

Coxeter, HSM; *Projective geometry*, 2nd edn, Springer Verlag (1974)

Hilbert, D & Cohn-Vossen, S; *Geometry and the imagination*, 2nd ed, Chelsea (1999)

Richeson, D; *Euler's Gem - The polyhedron formula and the birth of topology*, Princeton (2010)

Lakatos, I; *Proofs and refutations - The logic of mathematical discovery*, CUP (1976)

Grünbaum, B; "Are your polyhedra the same as my polyhedra?" *Discrete and comput. geom.: the Goodman-Pollack festschrift*, ed. B. Aronov et al, Springer (2003), pp. 461-488.

Grünbaum, B; "'New' uniform polyhedra", *Discrete geometry, in honor of W Kuperberg's 60th birthday: monographs and textbooks in pure and applied mathematics* **253**, Dekker, New York, 2003, pp. 331–350.

Grünbaum, B; "Graphs of polyhedra; polyhedra as graphs", Discrete mathematics **307**, 2007, 445 – 463

Stewart, B; *Adventures among the toroids*, self-published (1970).

Gailiunas, P & Sharp, J; "Duality of polyhedra", *Internat. journ. of math. ed. in science and technology*, **36**, No. 6 (2005), pp. 617-642.

Grünbaum, B & Shephard, G; "Duality of polyhedra", *Shaping space – a polyhedral approach*, ed. Senechal and Fleck, Birkhäuser (1988), pp. 205-211

Wenninger, M; *Dual models*, CUP, 1983

Coxeter, HSM et. al.; *The fifty-nine icosahedra"*, 3rd edn, Tarquin, 1999.

Bridge, NJ; "Facetting" the dodecahedron, *Acta Crystallographica* **A30** [1974), pp. 548-552.

Inchbald, G.; "Facetting diagrams", *The Mathematical Gazette*, **90**, No. 518, July 2006, pp.253-261.

Inchbald, G.; Morphic polytopes blocki page.

Recent changes

28 Sept 2017: Move morphic discussion to new page.

25 May 2017: Update some answers, expand and reorganise subsequent material.

4 October 2016: Starting point. Add "Facetting diagrams" ref.

19 July 2015: Correction to morphic discussion following embarrassing blunder. Other tidyings.

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