A Polyhedron Full of Surprises

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Let's become acquainted with our special polyhedron right away. In FIGURE 1 we see a convex polyhedron having 20 faces, 38 edges, and 20 vertices. It is a centrally symmetric polyhedron embedded in $\mathbb{R}^3$ with its center at the origin.

Any polyhedron with 20 faces can be called an icosahedron. But to avoid any possible confusion between it and other icosahedra, in particular the regular icosahedron, we shall refer to it as *Kirkman's Icosahedron* in honor of the British mathematician Thomas Penyngton Kirkman (1806–1895). It does not appear in his work, but as we will see, many of its properties are related to those he studied. Consult Biggs [**2**] for more details about his life and work, or read some of his own publications on polyhedra [**9, 10**].

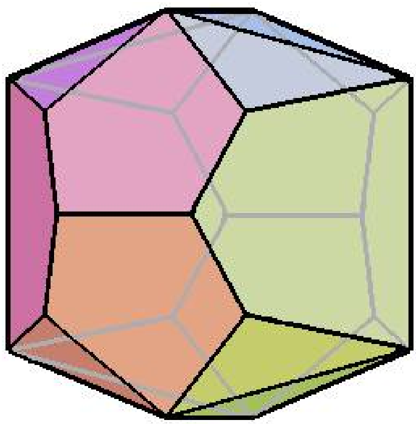

Figure 1 Kirkman's Icosahedron.

This polyhedron has many amazing properties. Let's have a look at them.

First set of surprises

We were able to construct it so that all twenty of its vertices have integer coordinates:

$$(\pm 9,\ \pm 6,\ \pm 6)$$
$$(\pm 12,\ \pm 4,\ 0)$$
$$(0,\ \pm 12,\ \pm 8)$$
$$(\pm 6,\ 0,\ \pm 12)$$

They are also indicated in FIGURE 2.

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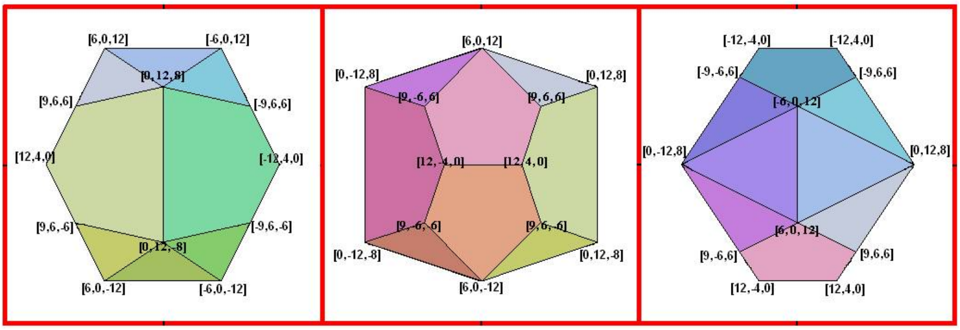

Figure 2 Three views of Kirkman's Icosahedron.

As for the lengths of its edges, it is easy to prove that they are all integers as well and their values are restricted to the numbers in the set

$$\{7, 8, 9, 11, 12, 14, 16\}.$$

A Schlegel diagram for Kirkman's Icosahedron, a projection of it onto the plane, is helpful because it allows us to display these lengths more clearly (see FIGURE 3).

Now, what about the volume of Kirkman's Icosahedron?

To compute it let us join the origin to all the vertices and edges of the object, splitting the polyhedron up into a collection of pyramids. The volume is then the sum of the volumes of these constituent pyramids. There are only four different kinds of pyramids: two with pentagonal bases and two with triangular bases. TABLE 1 is a summary of our findings. Each pyramid's volume is an integer!

From these results we finally get

$$4 \cdot 384 + 4 \cdot 576 + 4 \cdot 288 + 8 \cdot 192 = 6528$$

for the volume of Kirkman's Icosahedron.

Now, let us turn our attention to a completely different topic. What happens when we consider the traveling salesman problem on the surface of our polyhedron?

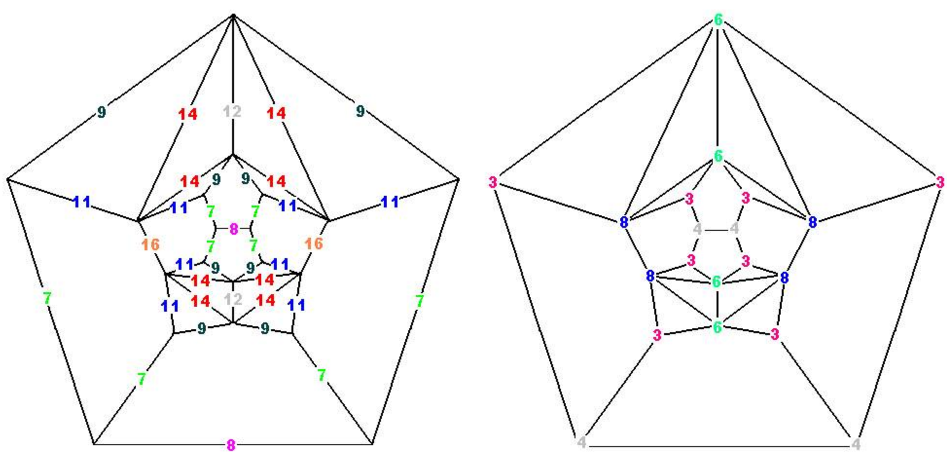

Figure 3 Schlegel diagram for Kirkman's Icosahedron, showing the lengths of the edges and corresponding weights for the vertices.

TABLE 1: Volumes.

	B	h	Volume
pentagon I	$48\sqrt{5}$	$24/\sqrt{5}$	384
pentagon II	$48\sqrt{13}$	$36/\sqrt{13}$	576
triangle I	$24\sqrt{10}$	$36/\sqrt{10}$	288
triangle II	$12\sqrt{17}$	$48/\sqrt{17}$	192

Second surprise

The study of cycles on polyhedra can be traced back to Kirkman's work. In [**9**] he considered the following question: Given the graph of a polyhedron, can we always find a cycle that passes through each vertex once and only once? The traveling salesman problem in addition takes the intermediate distances between the vertices into consideration, and asks to find a cycle of shortest possible length. All the information we need in order to solve this problem can be gathered from the Schlegel diagram in FIGURE 3. First we determine the number of possible closed tours (Hamiltonian cycles) for Kirkman's Icosahedron. With the help of a computer program, we find that there are precisely 206 of them. Next we need to solve the question: Which of these has the minimum total length? We could examine them one by one or use the following shortcut.

Suppose that, instead of using the lengths for the edges in our computations, we decide to attach certain weights $w(u)$ to the vertices u that we encounter in the Schlegel diagram. But how, exactly, are we supposed to assign these weights to the vertices? We choose them so that the length $c(u, v)$ of each edge (u, v) can alternately be computed as $c(u, v) = w(u) + w(v)$. It is a simple exercise to see that this can be done only in one way (see the right image of FIGURE 3).

Since an arbitrary Hamiltonian cycle H has to include every vertex of S exactly once, its overall length is just twice the sum of the weights of all the vertices:

$$\sum_{u,v\in H} c(u, v) = 2\sum_{u} w(u) = 192.$$

So all 206 cycles have the same total length!

In other words, for Kirkman's Icosahedron we get what is known as a *constant* traveling salesman problem (for additional information on this problem, consult [**8**]).

Next, let us briefly discuss a very interesting notion in the theory of polyhedra: duality. What can be said about the dual of Kirkman's Icosahedron?

Third surprise

We need to concentrate on two important concepts for polyhedra: isomorphism and duality, or *syntypicism* and *polar-syntypicism*, as they were referred to in one of Kirkman's very entertaining papers on polyhedra (see [**10**]). We consider these first from a combinatorial point of view.

When, in a typical book on polyhedra (take for instance [**7**]) an author says:

- a truncated square pyramid and a parallepiped are *isomorphic*,
- or a distorted convex prism and a squashed convex dipyramid are *dual* to each other,

what exactly does he or she mean by that? Grünbaum and Shephard [**4, 5**] and also Ashley et al. [**1**] suggest the following definitions:

- Two convex polyhedra P_1 and P_2 are *isomorphic* if there is a one-to-one correspondence between the family of vertices and faces of P_1 and the family of vertices and faces of P_2 which preserves inclusion.
- Two convex polyhedra P and P^* are said to be *duals* of each other if one can establish a one-to-one correspondence between the family of vertices and faces of P and the family of faces and vertices of P^* which reverses inclusion.

Note that these notions belong to the combinatorial or topological theory of convex polyhedra, because they both ignore any geometrical aspects.

Although we could easily find the dual of Kirkman's Icosahedron directly from the Schlegel diagram, we prefer to employ a more constructive approach involving the notion of polarity. It is an appropriate tool that will allow us to visualize a pair of dual polyhedra as an arrangement of interpenetrating solids. A familiar example involving the regular dodecahedron and its dual, the regular icosahedron, is shown in FIGURE 4. For additional pairs, see also [**7, 16**].

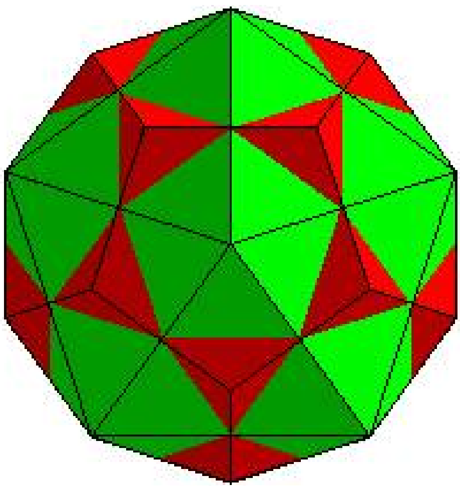

Figure 4 Compound of the regular dodecahedron and its dual, the regular icosahedron.

We would like to achieve something similar for a compound of Kirkman's Icosahedron and one of its duals. Therefore, we need to start by introducing the concept of the *polar set* of a set (see Lay [**11**]). If the inner product of $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ is given, as usual, by $x \cdot y = x_1y_1 + x_2y_2 + x_3y_3$, then the polar set K^* of any nonempty subset K of $\mathbb{R}^3$ is defined by

$$K^* = \left\{ y \mid \text{for all } x \in K,\ x \cdot y \leq r^2 \right\}.$$

Most authors require that $r = 1$ in the previous expression, but for our purposes it will be more convenient to use $r = 12$.

By means of two examples, let us show that polarity is a correspondence which relates each point to a plane and each plane to a point.

- When K consists of a single point, say $K = (x_1, x_2, x_3) = (9, 6, 6)$, then K^* is the closed half-space

$$9y_1 + 6y_2 + 6y_3 \leq 144,$$

which is bounded by the plane

$$9y_1 + 6y_2 + 6y_3 = 144.$$

- When K is a plane, say the one containing the pentagon with vertices $(9, 6, \pm 6)$, $(12, 4, 0)$, $(0, 12, \pm 8)$, then $K = \{(x_1, x_2, x_3) \mid 8x_1 + 12x_2 = 144\}$. In order to determine K^* we must find all those y for which

$$x_1 y_1 + x_2 y_2 + x_3 y_3 \leq 144 \quad \text{for all} \quad x \in K.$$

  It is easy to verify that in this case K^* is simply a ray (a half-line)

$$(y_1, y_2, y_3) = (8, 12, 0) + \lambda(-8, -12, 0) \quad \text{where } \lambda \geq 0$$

  having $(8, 12, 0)$ as its endpoint.

It should be clear that, in a completely analogous fashion, we can determine the polar sets of all the vertices and planes of Kirkman's Icosahedron (see TABLE 2 and TABLE 3).

TABLE 2: Vertices of P and their polar sets.

Coordinates of P	Face-planes of P^*
$(\pm 9, \pm 6, \pm 6)$	$\pm 9y_1 \pm 6y_2 \pm 6y_3 = 144$
$(\pm 12, \pm 4, 0)$	$\pm 12y_1 \pm 4y_2 = 144$
$(0, \pm 12, \pm 8)$	$\pm 12y_2 \pm 8y_3 = 144$
$(\pm 6, 0, \pm 12)$	$\pm 6y_1 \pm 12y_3 = 144$

TABLE 3: Faces of P and their polar sets.

Face-planes of P	Coordinates of P^*
$\pm 12x_1 \pm 6x_3 = 144$	$(\pm 12, 0, \pm 6)$
$\pm 8x_1 \pm 12x_2 = 144$	$(\pm 8, \pm 12, 0)$
$\pm 4x_2 \pm 12x_3 = 144$	$(0, \pm 4, \pm 12)$
$\pm 6x_1 \pm 6x_2 \pm 9x_3 = 144$	$(\pm 6, \pm 6, \pm 9)$

Our interest in the search for all the face-planes and coordinates of P^* is justified by the following basic result (see Lay [**11**]):

> If P is a bounded, convex polyhedron, which encloses the origin, then the polar set of P is itself a polyhedron P^* dual to P.

Before actually constructing the dual polyhedron P^*, we want to compare the coordinates for the vertices and also the equations for the face-planes of P and P^*. It is clear that

- whenever P has a vertex with coordinates (a, b, c), then P^* has a corresponding vertex with coordinates (c, b, a), and
- whenever P has a face-plane whose equation is $ax_1 + bx_2 + cx_3 = r^2$, then P^* has one given by $cy_1 + by_2 + ay_3 = r^2$.

This, in effect, means that by a rigid motion (reflection in the plane $x_1 = x_3$) we can bring the polyhedron P into coincidence with its dual P^*. Thus P and P^* are not just duals, but also isomorphic. They are in fact congruent. Hence duality and isomorphism occur simultaneously, a property that allows us to identify our polyhedron as being *self-dual* or *polar-syntypic with itself* or *autopolar* for short, as Kirkman would express it. For additional information on self-dual polyhedra, consult [**1, 4, 5, 10**]. In FIGURE 5 we have included three different images for the compound of Kirkman's Icosahedron and its dual.

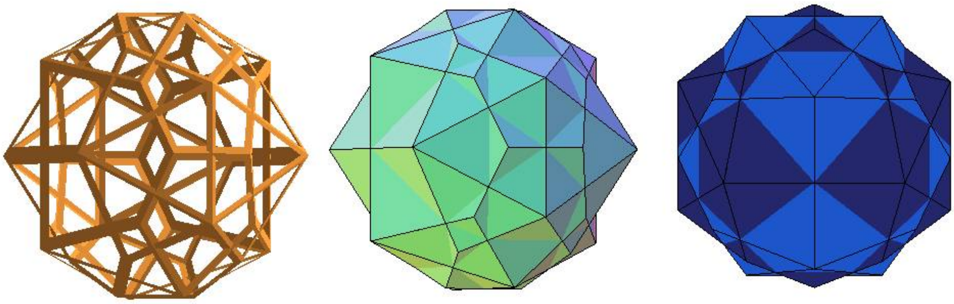

Figure 5 Compound of Kirkman's Icosahedron and its dual.

Closer inspection of the dodecahedron-icosahedron compound shown in FIGURE 4 reveals that the edges of both polyhedra intersect each other at right angles. It can be shown that they are, in fact, tangent to a so-called midsphere.

Does Kirkman's Icosahedron have a midsphere, and if so, what can we gain from it?

Fourth surprise

Let us consider two neighboring vertices of Kirkman's Icosahedron, say $P_0 = (9, 6, 6)$ and $P_1 = (6, 0, 12)$, and let M be the sphere with radius 12 centered at the origin. The line through the two points P_0 and P_1 is given by the parametric equation: $P(\lambda) = P_0 + \lambda(P_1 - P_0) = (9 - 3\lambda, 6 - 6\lambda, 6 + 6\lambda)$. Upon substitution into the equation for the sphere $x_1^2 + x_2^2 + x_3^2 = 144$, we get

$$9\lambda^2 - 6\lambda + 1 = (3\lambda - 1)^2 = 0.$$

Since this equation has two coincident real roots the line intersects the sphere in the unique point $(9 - 3\lambda, 6 - 6\lambda, 6 + 6\lambda)|_{\lambda=1/3} = (8, 4, 8)$, which means that the line is tangent to the sphere. Repeating this process for every pair of neighboring vertices, we obtain the set of contact points:

$$\left(\pm\frac{72}{7}, \pm\frac{36}{7}, \pm\frac{24}{7}\right) \qquad \left(\pm\frac{24}{7}, \pm\frac{36}{7}, \pm\frac{72}{7}\right)$$

$$\left(\pm\frac{72}{11}, \pm\frac{84}{11}, \pm\frac{72}{11}\right) \qquad (\pm 8, \pm 4, \pm 8)$$

$$(\pm 12, 0, 0) \qquad (0, \pm 12, 0) \qquad (0, 0, \pm 12)$$

So we conclude that all the edges of Kirkman's Icosahedron are tangent to the midsphere (see FIGURE 6). There is a "canonical" representation of this form for every

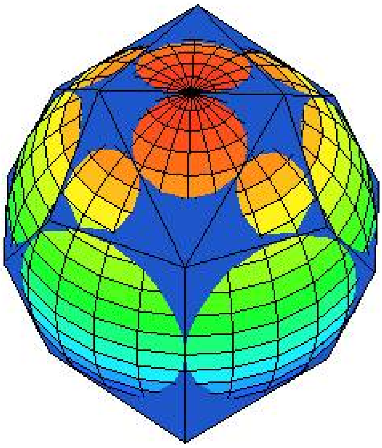

Figure 6 Kirkman's Icosahedron with midsphere.

polyhedron (Ziegler [**18**]). Moreover, in this case, the edges of the dual polyhedron are tangent to the sphere and, in fact, share the same edge-tangency (contact) points. In addition, the edges that correspond to each other under duality, intersect perpendicularly (for more details check Grünbaum [**6**], Sechelmann [**14**], and Ziegler [**17**]).

So, what are some of the consequences of our polyhedron having a midsphere M?

On the one hand, we immediately get two collections of non-overlapping circles on M's surface:

- the facet circles from the intersection of M with each of the faces of the polyhedron (referred to as a primal circle packing)
- the vertex horizon circles for each vertex (referred to as a dual circle packing).

A vertex horizon circle is the boundary of a spherical cap consisting of all those points on the sphere's surface which are visible from the respective vertex (see Sechelmann [**14**] and Ziegler [**18**]).

The circles from the primal circle packing touch if the corresponding faces are adjacent, and those from the dual circle packing touch if the corresponding vertices are adjacent (see FIGURE 7). Note that these two circle packings share the same edge-tangency points and that, moreover, they intersect orthogonally. This can be appreciated in FIGURE 7, which was obtained using the Koebe polyhedron editor developed by Sechelmann [**15**].

All those facet circles or incircles play an important role in the assignment of weights to the vertices that we used earlier in our treatment of the traveling salesman problem. Since two tangents drawn to a circle from an external point are always equal in length, the line segments from a vertex to the points where the incircle is tangent to the sides are congruent. Since the incircles touch if the corresponding faces are adjacent, we conclude that all the line segments from a vertex to each of its adjacent edge-tangency points are congruent. So this justifies assigning the weight $w(u) = |u - z|$ to the vertex u where z is any edge-tangency point on some incident edge.

Let us derive an additional consequence of having an assemblage of two interpenetrating polyhedra arranged around a midsphere. From the compound we are able to construct a pair of dual convex polyhedra, namely by considering

- the largest convex solid that is contained in it, and
- the smallest convex solid that contains it.

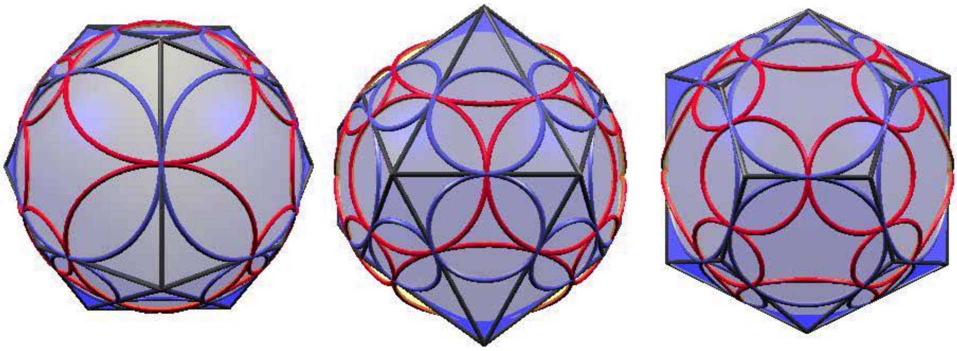

Figure 7 Kirkman's Icosahedron with facet (blue) and vertex horizon (red) circles.

Coxeter [**3**] refers to the first of these, which has the same face-planes as the compound as the core, and to the second, which has the same vertices as the compound as the case. The core and the case for the dodecahedron-icosahedron compound are well-known: They consist of the icosidodecahedron and the rhombic triacontahedron, respectively.

What can we say about the core and the case for the compound of Kirkman's Icosahedron and its dual? See FIGURE 8. First we remark that the core has all its vertices on the midsphere M, making it an inscribable polyhedron, whereas the case has all its faces tangent to the sphere M, making it a circumscribable polyhedron.

Second, since the vertices for the core are precisely the contact (edge-tangency) points and those for the case coincide with those of the compound, then it is fairly easy to see that we can get representations for both polyhedra having integral vertex coordinates!

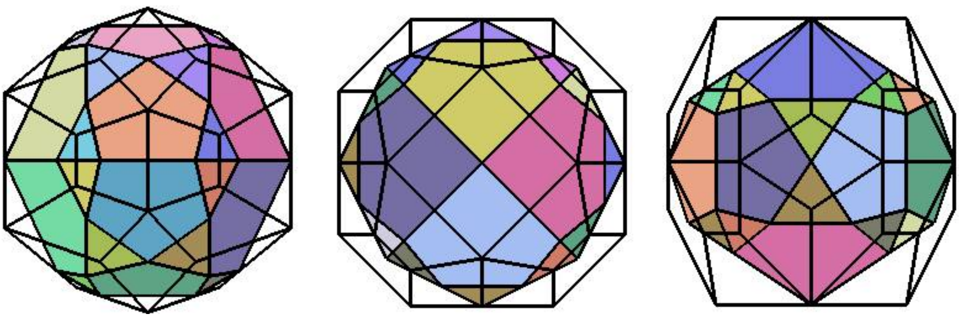

Figure 8 Case (transparent) and core (colored) for the compound of Kirkman's Icosahedron and its dual.

Finally, we want to consider a completely different question: Is this at all related with current research interests?

Fifth surprise

Surprisingly, we could say that the answer is yes. Richter [**12**], Rote [**13**], and Ziegler [**17**], for example, have been studying the problem of geometric realization for polyhedra which satisfy certain properties.

Interest has focused on the construction of convex polyhedra where either

(a) all of the vertex coordinates are small integers, or
(b) all of the edges are tangent to a sphere, or
(c) all of the edge lengths are integers.

In general, it is quite difficult to get such a polyhedral representation (see [**12, 13**]). On the other hand, Kirkman's Icosahedron satisfies all of these properties and some additional ones!

Considering that it all started with a simple straight-line drawing, the Schlegel diagram for the regular dodecahedron, a valid question is: How remarkable is Kirkman's Icosahedron?

REFERENCES

1. J. Ashley, B. Grünbaum, G. C. Shephard, and W. Stromquist, Self-duality groups and ranks of self-dualities, in *Applied Geometry and Discrete Mathematics,* The Victor Klee Festschrift, *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.* **4** (1991) 11–50.
2. N. L. Biggs, "T. P. Kirkman, mathematician," *Bull. London Math. Soc.*, **13**(2) (1981) 97–120. http://dx.doi.org/10.1112/blms/13.2.97
3. H. S. M. Coxeter, *Regular Polytopes*, 3rd ed., Dover, New York, 1973.
4. B. Grünbaum and G. C. Shephard, Some problems on polyhedra, *J. Geom.* **29**(2) (1987) 182–190. http://dx.doi.org/10.1007/BF01225208
5. B. Grünbaum and G. C. Shephard, Duality of polyhedra, in *Shaping Space: A Polyhedral Approach*, Birkhäuser, Boston, 1988, 205–211.
6. B. Grünbaum, Graphs of polyhedra; polyhedra as graphs, *Discrete Mathematics* **307** (2007) 445–463. http://dx.doi.org/10.1016/j.disc.2005.09.037
7. A. Holden, *Shapes, Space and Symmetry*, Dover, New York, 1991.
8. S. N. Kabadi, Polynomially solvable cases of the TSP, in *The traveling salesman problem and its variations*, *Comb. Optim.* **12**, Kluwer, Dordrecht, 2002, 489–583.
9. T. P. Kirkman, On the representation of polyedra, *Phil. Trans Roy. Soc. London* **146** (1856) 413–418. http://dx.doi.org/10.1098/rstl.1856.0019
10. T. P. Kirkman, On autopolar polyedra, *Phil. Trans. Royal Soc. London* **147** (1857) 183–216. http://dx.doi.org/10.1098/rstl.1857.0012
11. S. R. Lay, *Convex Sets and Their Applications*, Dover, New York, 2007.
12. J. Richter-Gebert, *Realization Spaces of Polytopes*, Lecture Notes in Mathematics **1643**, Springer-Verlag, Berlin, 1996. http://dx.doi.org/10.1007/BFb0093761
13. G. Rote, *Realizing Planar Graphs as Convex Polytopes*, Lecture Notes in Computer Science **7034**, Springer-Verlag, 2011. http://dx.doi.org/10.1007/978-3-642-25878-7_23
14. S. Sechelmann, *Discrete Minimal Surfaces, Koebe Polyhedra, and Alexandrov's Theorem. Variational Principles, Algorithms, and Implementation*, diploma thesis, Berlin, 2007.
15. S. Sechelmann, Koebe Polyhedron Editor, Deutsche Forschungsgemeinschaft, Polyhedral Surfaces-Software, http://www3.math.tu-berlin.de/geometrie/ps/software.shtml
16. M. J. Wenninger, *Dual Models*, Cambridge Univ. Press, Cambridge, 1983.
17. G. M. Ziegler, *Lectures on Polytopes*, Graduate Texts in Mathematics **152**, Springer-Verlag, New York, 1995.
18. G. M. Ziegler, Convex Polytopes: Extremal constructions and f-vector shapes, in *Geometric Combinatorics*, IAS/Park City Math. Ser. **13**, Amer. Math. Soc., Providence, RI, 2007, 617–691.

Summary The problem of geometric realization for convex polyhedra, which satisfy certain desirable properties, has received quite a bit of attention lately. Interest, mainly, has been on polyhedral representations where either all of the vertex coordinates are small integers, or all of the edge lengths are integers, or all of the edges are tangent to a sphere. In general, it is not easy to construct a convex polyhedron satisfying any of those criteria. We introduce a remarkable polyhedron that satisfies all of them.

HANS L. FETTER is a professor at the Universidad Autónoma Metropolitana in Mexico City. He is mainly interested and engaged in problems of a geometric nature: from those involving bounds for the dihedral angle-sum of polyhedra to those concerning space-filling, including of course self-reproducing polyhedra, all the way to the study of periodic orbits in billiards bounded by a smooth curve. The present article originated from the study of certain new self-dual polyhedra.