M.C.Escher Kaleidocycles

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The purpose of this text is to answer some questions that arise in connexion with kaleidocycles: What properties must tetrahedra have, so that continous and twistable rings can be built from them? How can the rotation of such a ring be described mathematically? For what number of tetrahedra exist kaleidocycles? In addition we want to briefly describe some special cases of kaleidocycles.

Regular kaleidocycles

First we restrict our considerations to kaleidocycles consisting of regular tetrahedra.

I. Let A, B, C, D be the vertices of a regular tetrahedron. Let P be the midpoint of the edge $[AB]$, and Q the midpoint of the edge $[CD]$. Furthermore let M be the midpoint of $[PQ]$ (then M is also the center of gravity of the tetrahedron). It holds:

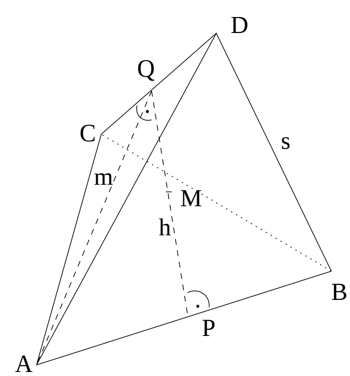
$$AB \perp PQ \perp CD \perp AB. \tag{1}$$

By s we denote the side length of the tetrahedron. Let m be the height of the faces (equilateral triangles). Let $h := \overline{PQ}$. Then we have

$$\left(\frac{s}{2}\right)^2 + h^2 = m^2 = s^2 - \left(\frac{s}{2}\right)^2$$

and it follows

$$s = h\sqrt{2}. \tag{2}$$



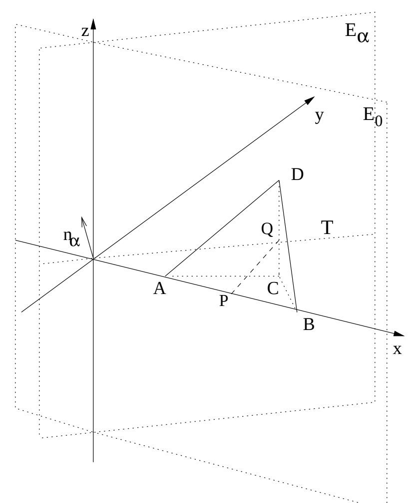
II. Let $n \in \mathbb{N}, n \geq 8$. Let

$$E_0 := \{(x, y, z) \in \mathbb{R}^3 \mid y = 0\}$$

be the x-z-plane. Let $\alpha := \frac{2\pi}{n}$ (then $0 < \alpha \leq \frac{\pi}{4}$ as $n \geq 8$) and let

$$\begin{aligned} E_\alpha &:= \{(x, y, z) \in \mathbb{R}^3 \mid y = x \tan\alpha\} \\ \vec{n}_\alpha &:= (-\sin\alpha, \cos\alpha, 0). \end{aligned}$$

The planes E_α and E_0 intersect in the z-axis. The angle between them is α. The vector $\vec{n}_\alpha$ is a normal vector to the plane E_α.



III. Suppose a regular tetrahedron T (notations as in I.) is positioned as follows:

i) A, B, P, Q lie in the x-y-plane

ii) $A, B, P \in E_0$

iii) $C, D, Q \in E_\alpha$

vi) C, D, Q have positive y-coordinate

Because of $0 < \alpha \leq \frac{\pi}{4}$ and (1) such a tetrahedron exists and is uniquely determined (for given h).
In addition A has positive x-coordinate, as

$$\overline{AP} = \frac{s}{2} = \frac{h}{2}\sqrt{2} < h \leq \frac{h}{\tan\alpha} = \frac{\overline{PQ}}{\tan\alpha} = \overline{OP}$$

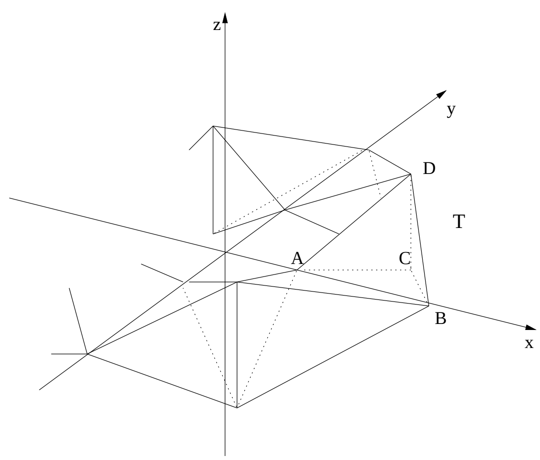
(because of $0 < \alpha \leq \frac{\pi}{4}$ we have $0 < \tan\alpha \leq 1$).
We also note:

$$\text{the vectors} \quad \overrightarrow{AB},\ \vec{n}_\alpha,\ \overrightarrow{CD} \quad \text{form a right-handed system,} \tag{3}$$

$$\text{the vectors} \quad \overrightarrow{AB},\ \overrightarrow{CD},\ \overrightarrow{QP} \quad \text{form a right-handed system} \tag{4}$$

In V. we will see that a tetrahedron positioned as above can be rotated around the axis PQ without violating conditions ii),iii) and iv) (the points P und Q, however, move within the planes E_0 and E_α then).

IV. Besides $n \geq 8$ let n be even. Reflecting the tetrahedron T about the plane E_α yields another tetrahedron T_2, that shares vertices C and D with T. By successively rotating T and T_2 about the z-axis by angle 2α further tetrahedra are obtained (altogether n tetrahedra) that form (because of n even and $\alpha = \frac{2\pi}{n}$) a closed ring (every two neighbouring tetrahedra share one common edge). This ring is called a <u>regular kaleidocycle</u>.



V. Now we show how a tetrahedron T can be rotated within the delimiting planes E_0 und E_α such that conditions ii),iii) and iv) from III. remain fulfilled. Then by symmetry it follows that a ring of tetrahedra assembled as in IV. can be inverted (while the property that neighbouring tetrahedra share one common edge is preserved).

We choose the parameter $t \in [0, 2\pi[$ to describe the rotation of the tetrahedron T in the sense that t specifies the actual angle between $\overrightarrow{AB}$ and the positive x-axis.
By $A_t, B_t, C_t, D_t, P_t, Q_t$ we denote the positions of the corresponding points at time $t \in [0, 2\pi[$.

Thus

$$\vec{u} := \frac{\overrightarrow{A_tB_t}}{\|\overrightarrow{A_tB_t}\|} = \begin{pmatrix} \cos t \\ 0 \\ \sin t \end{pmatrix} \quad \in E_0.$$

By (1), (3) we obtain ($\times$ denotes the cross product of vectors)

$$\begin{aligned} \vec{v} \quad &:= \quad \frac{\overrightarrow{C_tD_t}}{\|\overrightarrow{C_tD_t}\|} = \frac{1}{\|\vec{u} \times \vec{n}_\alpha\|}(\vec{u} \times \vec{n}_\alpha) \\ &= \quad \frac{1}{\sqrt{\sin^2 t + \cos^2 t \cos^2 \alpha}} \begin{pmatrix} -\sin t \cos \alpha \\ -\sin t \sin \alpha \\ \cos t \cos \alpha \end{pmatrix} = \frac{1}{\sqrt{1 + \sin^2 t \tan^2 \alpha}} \begin{pmatrix} -\sin t \\ -\sin t \tan \alpha \\ \cos t \end{pmatrix} \quad \in E_\alpha \end{aligned}$$

Furthermore let

$$\begin{aligned} \vec{w} \quad &:= \quad -(\vec{u} \times \vec{v}) \\ &= \quad \frac{1}{\sqrt{\sin^2 t + \cos^2 t \cos^2 \alpha}} \begin{pmatrix} -\sin^2 t \sin \alpha \\ \cos \alpha \\ \sin t \cos t \sin \alpha \end{pmatrix} = \frac{1}{\sqrt{1 + \sin^2 t \tan^2 \alpha}} \begin{pmatrix} -\sin^2 t \tan \alpha \\ 1 \\ \cos t \sin t \tan \alpha \end{pmatrix} \end{aligned}$$

(it is $\|\vec{w}\| = 1$).
By (1) and (4) it is $h\vec{w} = \overrightarrow{P_tQ_t} = Q_t - P_t$, which we may write as

$$h \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} - \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}.$$

Considering $P_t \in E_0$ and $Q_t \in E_\alpha$ (i.e. $p_2 = 0$ and $q_2 = q_1 \tan \alpha$) we obtain

$$q_2 = hw_2, \qquad q_1 = h\frac{w_2}{\tan \alpha}, \qquad p_1 = q_1 - hw_1 = h(\frac{w_2}{\tan \alpha} - w_1).$$

We require that the center M of $[P_tQ_t]$ always remains in the x-y-plane. As q_3 and w_3 have the same sign, it follows

$$q_3 = -p_3 = h\frac{w_3}{2}.$$

Altogether we have (with w as above)

$$P_t = h \begin{pmatrix} \frac{w_2}{\tan \alpha} - w_1 \\ 0 \\ -\frac{w_3}{2} \end{pmatrix} \quad \in E_0, \qquad Q_t = h \begin{pmatrix} \frac{w_2}{\tan \alpha} \\ w_2 \\ \frac{w_3}{2} \end{pmatrix} \quad \in E_\alpha$$

and A_t, B_t, C_t, D_t are given as

$$\begin{aligned} A_t &= P_t - \frac{h}{2}\sqrt{2}\vec{u}, \qquad & B_t &= P_t + \frac{h}{2}\sqrt{2}\vec{u}, \\ C_t &= Q_t - \frac{h}{2}\sqrt{2}\vec{v}, \qquad & D_t &= Q_t + \frac{h}{2}\sqrt{2}\vec{v}. \end{aligned}$$

In particular $A, B \in E_0$ and $C, D \in E_\alpha$.

VI. Another possibility to describe the position of the tetrahedron at time t is given by the affine transformation

$$\begin{aligned} \Phi_t \quad &: \quad \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \\ &\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} u_1 & w_1 & v_1 \\ u_2 & w_2 & v_2 \\ u_3 & w_3 & v_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + h \begin{pmatrix} \frac{w_2}{\tan\alpha} - \frac{w_1}{2} \\ \frac{w_2}{2} \\ 0 \end{pmatrix} \end{aligned}$$

By Φ_t all points of a tetrahedron, such that its center is the origin and such that

$$\frac{\overrightarrow{AB}}{\|\overrightarrow{AB}\|} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \frac{\overrightarrow{PQ}}{\|\overrightarrow{PQ}\|} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \frac{\overrightarrow{CD}}{\|\overrightarrow{CD}\|} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

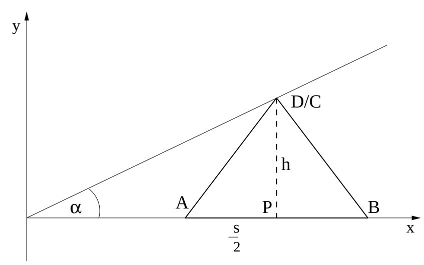
are mapped onto corresponding points of a tetrahedron that lies in the desired position for time t.

VII. We have assumed n even and $n \geq 8$. For $n \leq 6$ no twistable regular kaleidocycle exists:

Consider a kaleidocycle of n tetrahedra (n even) at time $t = 0$. Let p_1 be the x-coordinate of P. Obviously

$$p_1 \geq \frac{s}{2} = \frac{h}{2}\sqrt{2}$$

must hold, because otherwise several tetrahedra would intersect in the origin.



Now $p_1 = \frac{h}{\tan\alpha}$ and as $\alpha = \frac{2\pi}{n}$ we obtain the condition

$$\tan\frac{2\pi}{n} \leq \sqrt{2}$$

which for even n is only valid for $n \geq 8$. It follows that kaleidocycles consisting of regular tetrahedra must have at least 8 components in order to be rotatable. A regular kaleidocycle with 6 tetrahedra can be assembled, but it cannot be brought to the position $t = 0$ and therefore cannot be rotated completely. And that at least 6 tetrahedra are needed to build a kaleidocycle is obvious.

In the case $n = 6$, however, it is possible to have rotatable kaleidocycles, when irregular tetrahedra are used. More on that in the next section.

Normal kaleidocycles

VIII. Based on the previous section that dealt with regular kaleidocycles we show in the following how a whole class of kaleidocycles can be defined by introducing certain parameters.

We use the same notation as before. In particular let n be even and $n \geq 6$, and $\alpha = \frac{2\pi}{n}$.

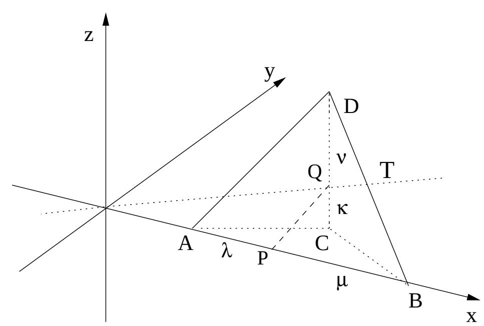
We have seen that the positions of the vertices A, B, C, D (we now omit the indices t) of a regular tetrahedron in a regular kaleidocycle are determined by the positions of the points P and Q as well as the vectors $\vec{u}$ and $\vec{v}$ (which in turn represent the directions of the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$):

$$A = P - \frac{h\sqrt{2}}{2}\vec{u}, \qquad B = P + \frac{h\sqrt{2}}{2}\vec{u},$$
$$C = Q - \frac{h\sqrt{2}}{2}\vec{v}, \qquad D = Q + \frac{h\sqrt{2}}{2}\vec{v}.$$

The normed vectors $\vec{u}$ and $\vec{v}$ were scaled with $\frac{h\sqrt{2}}{2}$, so that $ABCD$ is a regular tetrahedron.

If we instead set

$$A = P - \lambda\vec{u}, \qquad B = P + \mu\vec{u},$$
$$C = Q - \kappa\vec{v}, \qquad D = Q + \nu\vec{v}.$$



with arbitrary $(\lambda, \mu, \kappa, \nu) \in \mathbb{R}^4$, then $ABCD$ is still a (not necessarily regular) tetrahedron with

$$A, B \in E_0,$$
$$C, D \in E_\alpha.$$

By placing further tetrahedra that are equivalent to $ABCD$ in the same manner as in IV. we again obtain a closed ring where neighbouring tetrahedra share one common edge.

In order for such a kaleidocycle to be rotatable, we must have

$$|\lambda|, |\mu|, |\kappa|, |\nu| \leq \frac{h}{\tan\alpha}$$

(otherwise there are positions of the kaleidocycle for which several tetrahedra intersect in the origin, see VII.)

Within this restriction for the parameters there are still different configurations that essentially yield the same kaleidocycle (the configurations $(\lambda, \mu, \kappa, \nu)$, $(\kappa, \nu, \lambda, \mu)$ and $(\mu, \lambda, \nu, \kappa)$ for example are essentially the same). Therefore we further restrict the ranges of the parameters. It is easily seen that the following definition covers all essentially different configurations:

A kaleidocycle with n components that by symmetry is built from one tetrahedron $ABCD$ with

$$A = P - \lambda\vec{u}, \qquad B = P + \mu\vec{u},$$
$$C = Q - \kappa\vec{v}, \qquad D = Q + \nu\vec{v}.$$

where

$$\lambda, \kappa \in [0, \frac{h}{\tan\alpha}], \quad \mu \in [-\lambda, \lambda], \quad \nu \in [-\kappa, \kappa],$$

is called a normal kaleidocycle. Notation: $K_n(\lambda, \mu, \kappa, \nu)$.

Remark: By the definition of $P, Q, \vec{u}, \vec{v}$ tetrahedra that are components of normal kaleidocycles have the following (in our context crucial) property: two opposite edges AB and CD and their common perpendicular are pairwise orthogonal. (see (1)).

Special kaleidocycles

Using the parameters $n, \lambda, \kappa, \mu, \nu$ a varity of different forms and types of kaleidocycles can be designed. We want to conclude by mentioning some special configurations that are of interest because the corresponding kaleidocycles have additional geometrical properties.

IX. For $\lambda = \mu, \kappa = \nu$ we obain isosceles kaleidocycles, i.e. all faces of these kaleidocycles are isosceles triangles. This includes

- regular kaleidocycles with the configuration $n \geq 8, \lambda = \mu = \kappa = \nu = \frac{h}{2}\sqrt{2}$ (treated above),
- closed kaleidocycles with the configuration $\lambda = \mu = \kappa = \nu = \frac{h}{\tan\alpha}$, that have the property that at time $t = 0, t = \frac{\pi}{2}, t = \pi, t = \frac{3\pi}{2}$ vertices of several tetrahedra meet in the origin and thus the "eye" of the ring closes in these positions.

X. For $\mu = \nu = 0$ right-angled kaleidocycles are obtained, i.e. all faces are right triangles. Worth mentioning is the so called

- invertible cube defined by the configuration $n = 6, \lambda = \mu = \frac{h}{\tan\alpha}, \mu = \nu = 0$. The name comes from the fact that at time $t = \arccos\sqrt{\frac{2}{3}}$ this kaleidocycle becomes a cube by lengthening edges AB and CD (as well as corresponding edges of the other tetrahedra).